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Mathematical Reviews

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HISTORY

Bruins, E. M. Aperçu sur les mathématiques babyloniennes. Rev. Hist. Sci. Appl. 3, 301–314 (1950).

A general survey of problems and methods of Babylonian mathematics. The examples are taken from texts published by Neugebauer [Mathematische Keilschrifttexte, Springer, Berlin, 1935] and by Neugebauer and Sachs [Mathematical cuneiform texts, New Haven, 1945; these Rev. 8, 1], and from unpublished texts from Susa.

B. L. van der Waerden (Zurich).

Bruins, E. M. Quelques textes mathématiques de la mission de Suse. Nederl. Akad. Wetensch., Proc. 53, 1025–1033 = Indagationes Math. 12, 369–377 (1950).

Discussion of some mathematical cuneiform texts from Susa. The subjects treated in these texts are as follows. Text 4: Radius of circumscribed circle of an isosceles triangle. 3: Area of regular hexagon ($\sqrt{3}=7/4$) and heptagon (radius 35, side probably 30, area destroyed). *I*: Table of mathematical constants. *A* & *B*: Instructions for solving 2 linear equations with 2 unknowns. *C*: Computation of a rectangle with given ratio of sides and given diagonal by a similarity transformation. *D*: Another rectangle problem leading to a quadratic equation for l^4. *H*: Change of variables (technical term ki-gub). *Q*: Compound interest. *S*: Problem concerning division of a triangle by a line parallel to a side. A French translation is given for text *S* only; the exact wording of the other texts is not rendered.

B. L. van der Waerden (Zurich).

van der Waerden, B. L. Die Berechnung der ersten und letzten Sichtbarkeit von Mond und Planeten und die Venustafeln des Ammisaduqa. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 95, no. 1, 23–56 (1943).

Simplified and improved tables for the first and last visibility of the moon and the planets. *O. Neugebauer.*

Semyonov, L. Development of mathematical knowledge in Armenia in the seventh to thirteenth centuries. Armenian Affairs 1, 80–81 (1950).

Description of a text-book in arithmetic of Ananya Shirakatzi (7th century), whose manuscripts are now being studied in Armenia. An Armenian translation of Euclid's "Elements" by Gregory Magistros dates from the 11th century, though most of it has been lost. Another arithmetical text-book, also of the eleventh century, is by Hovhanes Sargavak. In it polygonal numbers are discussed.

D. J. Struik (Cambridge, Mass.).

Nobile, Vittorio. Il conflitto fra copernicisti e aristotelici nella sua essenza e nel pensiero di Galileo. I. Rilievi e precisazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 299–306 (1950).

Gericke, Helmuth. Zur Geschichte des mathematischen Denkens. Math.-Phys. Semesterber. 2, 71–97 (1951).

Severi, Francesco. La matematica russa: le sue tradizioni e i suoi progressi recenti. Archimede 2, 177–182 (1950).

Rosenthal, Arthur. The history of calculus. Amer. Math. Monthly 58, 75–86 (1951).

Gloden, A. Le développement de la théorie des séries depuis le début du 19e siècle jusqu'à nos jours. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. 19, 205–220 (1950).

Weyl, Hermann. 50 Jahre Relativitätstheorie. Naturwissenschaften 38, 73–83 (1951).

Lecture given at a meeting in October, 1950 of the Gesellschaft Deutscher Naturforscher und Ärzte.

Segre, Beniamino. Geometria dello spazio fisico. Rend. Sem. Mat. Fis. Milano 20 (1949), 26–36 (1950).

Expository account of the historical development of geometry as a theory of physical space.

Freudenthal, Hans. La première rencontre entre les mathématiques et les sciences sociales. Arch. Internat. Hist. Sci. (N.S.) 4, 25–34 (1951).

Boyer, Carl B. From Newton to Euler. Scripta Math. 16, 141–157, 221–258 (1950).

Simonart, Fernand. De Gauss à Cartan. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 1010–1025 (1950).

***Burgatti, Pietro. Memorie scelte.** Nicola Zanichelli, Bologna, 1951. viii+354 pp. (1 plate). 2500 Lire.

This volume contains 38 selected papers by Burgatti and a list of all his published mathematical works.

Dugas, René. Mécanisme cartésien. Revue Sci. 87, 195–204 (1949).

Faddeev, D. K. Boris Nikolaevič Delone (for his 60th birthday). Uspehi Matem. Nauk (N.S.) 5, no. 6(40), 159–163 (1950). (Russian)

Frankl', F. I. The hydrodynamical work of Euler. Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 170–175 (1950). (Russian)

***Oeuvres mathématiques d'Évariste Galois publiées en 1897, suivies d'une notice sur Évariste Galois et la théorie des équations algébriques par G. Verriest.** 2d ed. Gauthier-Villars, Paris, 1951. x+64+56 pp. (1 plate).

This volume contains two books originally published separately by Gauthier-Villars: Oeuvres mathématiques d'Évariste Galois, Paris, 1897; and G. Verriest, Évariste Galois et la théorie des équations algébriques, Paris, 1934.

Dubnov, Ya. S., and Raševskiĭ, P. K. V. F. Kagan: A short sketch of a scientific biography (for his 80th birthday). Trudy Sem. Vektor. Tenzor. Analizu 7, 16–30 (1949). (Russian)

★**Baumgardt, Carola. Johannes Kepler. Life and Letters. With an introduction by Albert Einstein.** Philosophical Library, New York, N. Y., 1951. 209 pp. (2 plates). $3.75.

Knopp, Konrad. Edmund Landau. Jber. Deutsch. Math. Verein. 54, Abt. 1, 55–62 (1951).

Keldyš, M. V. For the 50th birthday of Mihail Alekseevič Lavrent'ev. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 3–8 (1 plate) (1951). (Russian)

A list of Lavrent'ev's mathematical papers is included.

Hofmann, Jos. E. Der junge Newton als Mathematiker (1665–1675). Math.-Phys. Semesterber. 2, 45–70 (1951).

Tomonaga, Sin-itiro. Dr. Yoshio Nishina, his sixtieth birthday. Progress Theoret. Physics 5, i–ii (1 plate) (1950).

Pastori, Maria. Obituary: Giulio Vivanti. Rend. Sem. Mat. Fis. Milano 20 (1949), xv–xix (1 plate) (1950).

Armellini, Giuseppe. Vito Volterra e la sua opera scientifica. Ricerca Sci. 21, 3–12 (1951).

FOUNDATIONS

Hermes, Hans. Zum Begriff der Axiomatisierbarkeit. Math. Nachr. 4, 343–347 (1951).

The author calls a set of statements M syntactically axiomatizable if and only if there exists a finite subset M_0 and a finite set of decidable relations $R_i(a_1, \cdots, a_{n_i})$ (rules of inference) such that M is the closure of M_0 under the relations R_i. He shows that for any such system M there exists a single rule of inference such that taking any statement of M as axiom the system M is generated. The result is a corollary of the following: For any recursively enumerable set M there exists a recursive relation $R(a, b)$ such that the closure of the set consisting of any single element of M is M. If $\varphi(y)$ enumerates M, then $R(a, b)$ is defined as $(\exists x)(x \leqq \max(a, \mu y(\varphi(y) > a)) \wedge b = \varphi(x))$. *D. Nelson.*

Wang, Hao. Remarks on the comparison of axiom systems. Proc. Nat. Acad. Sci. U.S.A. 36, 448–453 (1950).

This paper concerns systems S with an arithmetized syntax. For such a system S, Con(S) is the arithmetical proposition expressing the consistency of S. A system S is translatable into a system S' if there exists a general recursive function T mapping the numbers representing the propositions of S into the corresponding numbers for S' such that T is a homomorphism with respect to the property of being a theorem and the operation of negation. Two systems S and S' are said to be of equal strength if Con (S) and Con (S') are mutually derivable in number theory; if Con (S) is derivable from Con (S') but not conversely, then S' is said to be stronger than S. A mathematical system is one which contains number theory; an elementary system one such that Con (S) is derivable in number theory. The following theorems are then said to follow by standard methods. If S is translatable into S', Con (S) is derivable from Con (S'); all consistent and decidable systems are elementary, but there exist elementary systems which are not decidable; every system S is translatable into the system formed by adjoining Con (S) to number theory. The author then postulates a system L such that number theory and the theory of classes of natural numbers can be obtained in it. The system L'' is obtained from L by adding variables of a higher type; L' is L'' with the restriction that the new variables are not allowed in the definitions of classes of lower types, L^* is L+Con (L). Then Con (L^*) is a theorem of L''; L' is translatable into L^*; there exists in L' a Tarski truth set for L and L^*; if one uses the same basic definitions in L' as in L and L^*, the principle of induction is independent of the axioms of L' and Con (L) is not provable in L'; with suitable definitions of the natural numbers, Con (L), but not Con (L^*) is provable in L', a truth definition for L cannot be obtained in L^*. These theorems illuminate the connections between the following relations between S and S': definability of a truth set for S in S', derivability of Con (S) in S', and translatability of S into S'.

H. B. Curry (Brussels).

Götlind, Erik. A system of postulates for Lewis's calculus $S1$. Norsk Mat. Tidsskr. 32, 89–92 (1950).

The author shows that if one takes as primitive rules (1) the rule of replacement of equivalents, (2) the rule of substitution, (3) the rule of adjunction, and (4) modus ponens, then the axiom-schemas ($B2$) $pq<p$, ($B3$) $p<pp$, ($B4'$) $(pq)r<(rq)p$, ($B6$) $(p<q)(q<r)<(p<r)$, ($B7$) $p(p<q)<q$, are sufficient for the Lewis algebra $S1$. Further, the last four schemes can be combined into a single scheme. By means of simple matrices the author shows that ($B4'$) and ($B2$) are each independent of the remaining schemas.

H. B. Curry (Brussels).

Hasenjaeger, Gisbert. Über eine Art von Unvollständigkeit des Prädikatenkalküls der ersten Stufe. J. Symbolic Logic 15, 273–276 (1950).

The author gives an example of two formulas which are always simultaneously valid or simultaneously invalid, but nevertheless are not deducible one from the other. The formulas are the negations of two formulas $\mathfrak{F}$ and $\mathfrak{G}$ considered by Hilbert and Bernays [Grundlagen der Mathematik, vol. 1, Springer, Berlin, 1934]. *H. B. Curry.*

Kalmár, László. Another proof of the Gödel-Rosser incompletability theorem. Acta Sci. Math. Szeged 12, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars A, 38–43 (1950).

Gödel [Monatsh. Math. Phys. 38, 173–198 (1931)] showed that every postulate system satisfying certain conditions contains an undecidable proposition. Rosser [J. Symbolic Logic 1, 87–91 (1936)] showed that a condition of ω-consistency arising in the Gödel theorem could be replaced by ordinary consistency. The author gives here another proof of this Rosser theorem. As in previous proofs of the Gödel theorem [C. R. Acad. Sci. Paris 229, 963–965, 1047–1049 (1949); these Rev. 11, 411] the author deduces the theorem by a quite easy argument, not "making use of the

deductive structure of the postulate system", from certain general conditions on the system. It is not clear, and it is not claimed, that there are systems satisfying his conditions which are not included in the original proof. But, even though the verification of some of these conditions may be somewhat laborious, a proof constructed along the lines indicated here promises to involve rather less technicality, for the usual special cases, than that in the original.

H. B. Curry (Brussels).

Schmidt, Arnold. Systematische Basisreduktion der Modalitäten bei Idempotenz der positiven Grundmodalitäten. Math. Ann. **122**, 71–89 (1950).

Students of modal logic have long been interested in the so-called "modalities": i.e., in the sentential functions of one variable which can be built up from the primitive functions "is false", "is possible", and "is necessary". Investigation of the relations between the various modalities, however, has hitherto been conducted within a whole system of modal sentential calculus, which allows also a symbolism for binary sentential connectives such as conjunction. [For results of this sort, referred to the various Lewis calculi, see Parry, J. Symbolic Logic **4**, 137–154 (1939); McKinsey ibid. **5**, 110–112 (1940); these Rev. **1**, 131; **2**, 66.] The present paper, on the other hand, is concerned with the investigation of modalities as such: i.e., the author develops a theory for the investigation of modalities which does not depend on the properties of any binary connectives. (In its present form, the theory enables one to deal only with the relation of equivalence between modalities; but the author indicates that in a later work he will take up problems of "implication" between modalities.) This theory is concerned throughout with the manipulation of finite words composed of the six symbols: z, $\bar{z}$, m, $\bar{m}$, g, and $\bar{g}$. These are to be interpreted intuitively as meaning "it is true that", "it is false that", "it is possible that", "it is impossible that", "it is necessary that", and "it is not necessary that", and words composed of them are to be read by concatenating the readings of the six symbols: Thus the word $\bar{m}zg$ is to be read "it is impossible that it is true that it is necessary that". It is supposed throughout that the operation of concatenation of words is associative, and that the relation of equivalence (symbolized by "$=$") is transitive and symmetric, and that equivalent words can be concatenated, on either the left or right, by the same word. (The author apparently intends for equivalence also to be reflexive, though this is not explicitly mentioned.)

In terms of this symbolism the author gives a careful and detailed investigation of the relation between various systems of axioms (equations connecting the six primitive modalities). Among other results, it is shown that there are only two interesting systems in which every modality is equivalent to a modality of first degree (each of these systems has just six irreducible modalities); and that there are only three additional interesting systems in which every modality is equivalent to a modality of second degree (one of these systems has eight irreducible modalities, and each of the other two has ten).

J. C. C. McKinsey.

Bergmann, Gustav. Concerning the definition of classes. Mind **60**, 95–96 (1951).

For the contextual definition of $\hat{z}f(z)$, the class of all z such that $f(z)$, Carnap [Meaning and Necessity, University of Chicago Press, 1947; these Rev. **8**, 430] has proposed that rather than allow $F(\hat{z}(f(z)))$ to stand for $(\exists g)(g\equiv f\cdot F(g))$ as Russell has done [Whitehead and Russell, Principia mathematica, v. 1, 2d ed., Cambridge University Press, 1925], we let the expression abbreviate $(g)(g\equiv f\supset F(g))$. The author objects to this on the ground that it has the consequence that everything which can be predicated of a class can then also be predicated of every property to which it corresponds, that is, $F(\hat{z}(f(z)))\supset F(f)$ becomes a theorem. It might be observed also that in a system in which equality is introduced as in the Principia, the law of identity for classes $\hat{z}(f(z))=\hat{z}(f(z))$ cannot be proved. This violates one of the requirements which Russell desired of a class notation.

D. Nelson (Washington, D. C.).

Ridder, J. Formalistische Betrachtungen über intuitionistische und verwandte logische Systeme. IV. Nederl. Akad. Wetensch., Proc. **53**, 1375–1389 = Indagationes Math. **12**, 445–459 (1950).

The author shows that certain axiom schemata are eliminable or may be weakened in the formulation of systems of constructive logic given in part III of this work [same vol., 787–799 = Indagationes Math. **12**, 231–243 (1950); these Rev. **12**, 71]. The principle result is a detailed demonstration of the Gentzen Hauptsatz [Math. Z. **39**, 176–210, 405–431 (1934)] for the weakest of these systems.

D. Nelson (Washington, D. C.).

Myhill, John R. A complete theory of natural, rational, and real numbers. J. Symbolic Logic **15**, 185–196 (1950).

The author presents a weaker version of Fitch's basic logic [same J. **7**, 105–114 (1942); these Rev. **4**, 125]. The system, containing free variables, is based on primitives to be interpreted as conjunction, disjunction, existential quantification, equality, ordered pair, class membership, an ancestral relation, and the natural number 0. A truth definition for the system is given, and consistency and completeness are established in the sense that a formula is provable if and only if it is true. The author shows that any general recursive relation may be represented in the system. Both classical negation and universal quantification are shown to be undefinable in the system, the introduction of these sacrifices completeness. Definitions of rational and real numbers are given, and the theory of these real numbers is to be developed in a sequel. The theory of real numbers will presumably show deficiencies similar to those noted by Specker [ibid. **14**, 145–158 (1949); these Rev. **11**, 151] for the system of Goodstein [Proc. London Math. Soc. (2) **48**, 401–434 (1945); these Rev. **8**, 245].

D. Nelson.

Ackermann, Wilhelm. Konstruktiver Aufbau eines Abschnitts der zweiten Cantorschen Zahlenklasse. Math. Z. **53**, 403–413 (1951).

The author describes a construction of a segment of the second number class. First he considers the set of all symbols such that (i) 1 is in the set, (ii) any other symbol belongs to this set if and only if it is of the form $\alpha+\beta$ or (α, β, γ), where α, β, γ are in the set. He next defines an ordering relation $<$ between symbols of this set. From these, the symbols which represent ordinals are chosen to be: (a) 1; (b) any sum of a (finite) decreasing sequence of symbols which represent ordinals; (c) any triple of symbols which represent ordinals. It is then shown that these really do form a model of a segment of the ordinals and that transfinite induction can be applied. For every limit ordinal α in this segment it is possible to define a uniquely determined in-

creasing sequence whose limit is α. The segment thus constructed is rather extensive and includes ϵ-numbers.
I. L. Novak (Berkeley, Calif.).

★Bolzano, Bernard. Paradoxes of the Infinite. Translated from the German of the posthumous Edition by Dr. Fr. Přihonský and furnished with a historical introduction by Donald A. Steele. Routledge and Kegan Paul, London, 1950. ix+189 pp.

Kreisel, G. Some remarks on the foundations of mathematics. An expository article. Math. Gaz. **35**, 23–28 (1951).

van der Waerden, B. L. Concerning space. Euclides, Groningen **26**, 207–219 (1951). (Dutch)

Lecture given at the University of Amsterdam, December, 1950.

Margenau, Henry, and Compton, John. Report on recent developments in the philosophy of quantum mechanics. Synthese **8**, 260–271 (1951).

★Esnault-Pelterie, Robert. Dimensional Analysis and Metrology. (The Giorgi System). Editions F. Rouge & Co., Lausanne, 1950. xiv+112 pp.

The author insists that, in the Giorgi system of electromagnetic units, only three (and not four) should be taken. His argument is based on the terminology and philosophical ideas of his earlier work [L'analyse dimensionelle, Rouge, Lausanne, 1948]; it is contrary to the usual view [P. Bridgman, Dimensional analysis, revised ed., Yale University Press, 1931, p. 102; G. Birkhoff, Hydrodynamics . . ., Princeton University Press, 1950, pp. 88–89; these Rev. **12**, 365] that the choice and even number of "fundamental" units is partly arbitrary. *G. Birkhoff.*

★Langhaar, Henry L. Dimensional Analysis and Theory of Models. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. xi+166 pp. (1 plate). $4.00.

Though primarily a text for senior and first-year graduate engineering students, this book has interest for applied mathematicians, as an introduction to engineering applications of dimensional analysis. It gives a clear and authentic picture of how dimensional analysis is currently used by mechanical engineers to set up model experiments and to derive useful formulas. (Its discussion of "astronomical units" in examples 2, 3, 7 seems less authentic.) In no other comprehensive treatment, is the basic engineering reasoning so clearly explained; over twenty applications are discussed in some detail. However, the rational basis for dimensional analysis is not adequately treated. The critical reader will agree with Bridgman [Dimensional analysis, revised ed., Yale University Press, 1931, p. 50] that physical experiment, and not the "philosophical insight" appealed to by the author (page v), is what enables one to "name the variables" relevant to phenomena. He will note that, by deprecating (in chapter X) the inspectional analysis of differential equations, the author takes away the theoretical basis for dimensional analysis [cf. G. Birkhoff, Hydrodynamics . . ., Princeton University Press, 1950, ch. 3, §10 ff.; these Rev. **12**, 365]. He will find the author's proof of the Pi theorem, taken from his paper [J. Franklin Inst. **242**, 459–463 (1946); these Rev. **8**, 247] less sharp and general than those of A. Martinot Lagarde [Analyse dimensionelle. Applications à la mécanique des fluides, Lille, 1946] or the reviewer [op. cit., ch. 3, §§1–7]. Most serious, the serious difficulties involved in interpreting model data are glossed over. Thus the informed reader will contrast the simple fig. 1 with the complex fig. 201 of Goldstein's Modern Developments in Fluid Dynamics [Oxford, 1938]; he will question the neglect of gravity and turbulence in the discussion of underwater explosion models; he will wonder what friction "law" is advocated in interpreting ship resistance model tests; and he may doubt whether scour and shoaling can be confidently predicted for many years, on the basis of existing model experiments.
G. Birkhoff (Cambridge, Mass.).

ALGEBRA

Robinson, H. F., and Watson, G. S. An analysis of simple and triple rectangular lattice designs. North Carolina Agricultural Experiment Station, Tech. Bul. No. **88**, 56 pp. (1949).

This is a thorough exposition of the methods of analysis of the designs in the title, in particular, the triple rectangular lattices introduced by Harshbarger [Biometrics **5**, 1–13 (1949); these Rev. **11**, 3]. The authors give many examples explaining the methods of analysis using inter- and intra-block information. A large variety of designs is tabulated in an appendix. *H. B. Mann* (Columbus, Ohio).

Škrášek, J. Application des méthodes mathématiques à la théorie des classifications. Publ. Fac. Sci. Univ. Masaryk no. **316**, 39 pp. (1 plate) (1949). (French. Czech summary)

A construction is given for producing two complementary classifications of arbitrary sets such as might serve for example in resolving ambiguous assignments of plants in biology. The classifications can be represented as trees like those considered by Ore [Duke Math. J. **9**, 573–627 (1942); these Rev. **4**, 128], on whose work the author leans.
J. Riordan (New York, N. Y.).

Bruck, R. H. Finite nets. I. Numerical invariants. Canadian J. Math. **3**, 94–107 (1951).

A net N of degree K order n is a system of points and lines such that: (1) N contains K nonempty classes of lines; (2) two lines a, c of N belonging to distinct classes intersect in exactly one point; (3) each point of N is on exactly one line of each class; (4) at least one line of N has n distinct points. A net is thus equivalent to $K-2$ orthogonal $n\times n$ Latin squares. Now let $f(P)$ be an integral single-valued function defined over the points P of the net N. The integer m is said to be represented by N if $f(P)$ sums to m over each line of N. The integer m is said to be represented positively by N if it can be represented by a nonnegative function $f(P)$. The smallest number represented by N is denoted by $\varphi(N)$. It is the greatest common divisor of all integers represented by N. The numbers $K-1$ and n are represented by N. The number 1 is positively represented if and only if a line can

be adjoined to N (transversal to all lines of N). Almost all multiples of $\varphi(N)$ are represented by N. The integer m can be represented by N if it can be represented by N mod n. Further results on $\varphi(N)$ are obtained which are too complicated to be stated in a review. The author further considers a direct product $N_1\times N_2$ of two sets N_1, N_2. The points of $N_1\times N_2$ are ordered pairs (P_1, P_2) with $P_1\subset N_1$, $P_2\subset N_2$. The lines are given by those pairs whose "coordinates" form lines of N_1, N_2 respectively. The author derives several interesting relations between $\varphi(N_1)$ and $\varphi(N_1N_2)$. A homomorphic mapping of a net N on a net N^1 is a single-valued exhaustive mapping that preserves incidence. The author proves various theorems about homomorphisms connecting also $\varphi(N)$ and $\varphi(N^1)$. In the last part of the paper the function $\varphi(N)$ is evaluated for nets of degree three or Latin squares. *H. B. Mann* (Columbus, Ohio).

Kurepa, Georges. Sur une définition et une ordination des nombres complexes. Bull. Soc. Math. Phys. Serbie 2, nos. 1–2, 1–18 (1950). (Serbo-Croatian. French summary)

This paper contains the definitions of (a) a group, (b) the complex numbers as ordered pairs of real numbers, (c) the usual product partial ordering of the complex numbers. It is proved, among other things, that the complex numbers are a field. *E. Hewitt* (Uppsala).

Kurepa, Georges. Sur la définition et l'ordination de l'ensemble des nombres complexes. Acad. Serbe Sci. Publ. Inst. Math. 3, 89–99 (1950).

A translation, with only minor changes, of the paper reviewed above. *E. Hewitt* (Uppsala).

Jou, Yuh-Lin. The "fundamental theorem of algebra" for Cayley numbers. Acad. Sinica Science Record 3, 29–33 (1950). (English. Chinese summary)

A polynomial in Cayley numbers is a sum of a finite number of monomials, each of which is parenthesized because of nonassociativity. It is proved that if such a polynomial has only one term of highest degree n, then it has at least one zero. The proof is topological, employing the degrees of homotopic mappings, and is similar to the method used to obtain the corresponding result for quaternions by Eilenberg and Niven [Bull. Amer. Math. Soc. 50, 246–248 (1944); these Rev. 5, 169] who had commented that their proof could be extended to Cayley numbers. *I. Niven.*

Richard, Ubaldo. Algebra e analisi nel teorema fondamentale dell'algebra. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 11–20 (1950).

Expository article on the use of nonalgebraic concepts in the various proofs of the so-called fundamental theorem of algebra, with particular reference to what constitutes an "algebraic" procedure. *I. Niven* (Eugene, Ore.).

Sce, Michele. Su alcune proprietà delle matrici permutabili e diagonalizzabili. Rivista Mat. Univ. Parma 1, 363–374 (1950).

Various theorems, mainly known, concerning matrices which can be transformed to diagonal form (by means of an orthogonal or a general transformation) are discussed. Further, conditions are found which are necessary and sufficient in order that $AB^r = C^rA$, r a positive integer >1, implies $AB=CA$. These are used to derive again some properties concerning normal matrices found by Wiegmann [Duke Math. J. 15, 633–638 (1948); these Rev. 10, 230]. *O. Todd-Taussky* (Washington, D. C.).

Parker, W. V. Characteristic roots and field of values of a matrix. Bull. Amer. Math. Soc. 57, 103–108 (1951).

This paper reports an address delivered in June, 1950, and surveys results published elsewhere by the author and others, mainly since 1939 when Browne [Amer. Math. Monthly 46, 252–265 (1939)] made a similar survey. There is a bibliography of twenty-five items which does not, however, include any reference to the work (some very recent) of L. Collatz [Eigenwertaufgaben mit technischen Anwendungen, Akademische Verlagsgesellschaft, Leipzig, 1949; these Rev. 11, 137] and H. Wielandt [Arch. Math. 1, 348–352 (1949); Math. Ann. 121, 234–241 (1949); Math. Nachr. 2, 328–339 (1949); Naturforschung und Medizin in Deutschland 1939–1946, Band 2, pp. 85–98, Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948, see pp. 97–98; these Rev. 11, 4, 307, 729, 114]. *W. Givens.*

Reid, William T. A note on the characteristic polynomials of certain matrices. Proc. Amer. Math. Soc. 1, 584–585 (1950).

Let A, B, C, and D be matrices with elements in an arbitrary field. Assume that A is an $m\times n$ matrix, B and C are $n\times m$ matrices, and D a square matrix of order m. W. V. Parker [Bull. Amer. Math. Soc. 55, 115–116 (1949); these Rev. 10, 424] proved that if ACA is the zero matrix, then AB and $A(B+C)$ have the same characteristic equations for every B. The author proves that the condition $ACA=0$ is not only sufficient, but also necessary. He generalizes this result as follows. In order that AB and $AB+D$ have the same characteristic equation it is necessary and sufficient that D be nilpotent and DA be the zero matrix. ~~(In the formulation of this theorem in the paper the letter G must be replaced by B.)~~ *A. Brauer* (Chapel Hill, N. C.).

Lidskiĭ, V. B. On the characteristic numbers of the sum and product of symmetric matrices. Doklady Akad. Nauk SSSR (N.S.) 75, 769–772 (1950). (Russian)

Let A and B be real symmetric matrices with characteristic roots $\lambda_1\geqq\cdots\geqq\lambda_n$ and $\mu_1\geqq\cdots\geqq\mu_n$. Let M be the set of all possible characteristic roots $\sigma_1\geqq\cdots\geqq\sigma_n$ of $A+B$. Let K_a (K_b) be the convex hull of the $n!$ points obtained by adding the λ's (μ's) to a permutation of the μ's (λ's); let $L=K_a\cap K_b$. The author proves that M is contained in L. The proof is by induction on the size of the matrix, and uses the reduction to diagonal form, and the exponential representation of orthogonal matrices. If the λ's and μ's satisfy certain inequalitities, it is shown that $M=L$. Similar results are indicated for the product of positive definite matrices. *I. Kaplansky* (Chicago, Ill.).

Wielandt, Helmut. Lineare Scharen von Matrizen mit reellen Eigenwerten. Math. Z. 53, 219–225 (1950).

Let $\mathfrak{M}$ be a family of $n\times n$ matrices with complex elements. For $\mathfrak{M}$ to be similar to the set of all Hermitian matrices, it is necessary and sufficient that (1) $\mathfrak{M}$ be closed with respect to real linear combination, (2) the elements of $\mathfrak{M}$ have only real eigenvalues, and (3) $\mathfrak{M}$ contain n^2 matrices which are linearly independent with respect to complex scalars. If (3) is relaxed to require only (3′) that $\mathfrak{M}$ be of dimension n^2 (which is the maximum under (1) and (2)) for real scalars, then $\mathfrak{M}$ is similar to the matrices in a triangular form (determined by a partition of n) with Hermitian blocks on the diagonal and zeros above. Under (1) and the requirement (3″) that the real dimension d of $\mathfrak{M}$ is $\geqq n^2$, (2) is equivalent to (2′) trace $\mathfrak{M}^2\geqq 0$ and (2″)

trace M^2 is real for all M in $\mathfrak{M}$. An example shows that (2″) cannot be replaced by "trace M is real." To prove the above, the author first shows that from (1) and (3″) it follows that (a) trace M^2 always real implies $d=n^2$ and $\mathfrak{M}$ consists of all A such that trace AM is real for all $M\epsilon\mathfrak{M}$; (b) if also trace M^3 is always real, $M\epsilon\mathfrak{M}$ implies $M^k\epsilon\mathfrak{M}$ for $k=2, 3, \cdots$; and, (c) if (a) and (b) hold and $D\epsilon\mathfrak{M}$ for a D which is diagonal with distinct nonzero eigenvalues, then $M=(m_{\mu\nu})\epsilon\mathfrak{M}$ implies $(m_{\mu\nu}E_{\mu\nu}+m_{\nu\mu}E_{\nu\mu})\epsilon\mathfrak{M}$ where $E_{\mu\nu}$ has zero components except for a one in the place (μ, ν).

W. Givens (Knoxville, Tenn.).

Drazin, M. P. A reduction for the matrix equation $AB=\epsilon BA$. Proc. Cambridge Philos. Soc. **47**, 7–10 (1951).

If A and B are square matrices of order n and $AB=\epsilon BA$ for a scalar ϵ, a similarity transformation (over a field containing the elements and roots of A and B) will simultaneously reduce A and B to $\left(\begin{smallmatrix} S & X \\ 0 & A_r \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} T & Y \\ 0 & B_r \end{smallmatrix}\right)$, where S and T are triangular of order r, and A_r and B_r are nonsingular of order $n-r$ with $0\leqq r\leqq n$. For $\epsilon=1$, one can require $r=n$ so A_r, B_r, X, and Y are not present and A and B are similar to triangular matrices. If $\epsilon\neq 1$, ST and TS are nilpotent and S and T have at least r zero eigenvalues between them. For $\epsilon\neq 1$ and $n-r\geqq 2$, ϵ must be a primitive kth root of unity where k divides $n-r$. Then A_r can be reduced to the direct sum $\operatorname{diag}(a, \epsilon a, \cdots, \epsilon^{k-1}a)$ and the most general form of B_r is

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & b_1 \\ b_2 & 0 & \cdots & 0 & 0 \\ \cdot & b_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & b_k & 0 \end{pmatrix}$$

where $b_ia=ab_i$ and the b_i are nonsingular but otherwise arbitrary. *W. Givens* (Knoxville, Tenn.).

Wilson, R. L. A method for the determination of the Galois group. Duke Math. J. **17**, 403–408 (1950).

Suppose the base-field F and the polynomial equation $p(x)=x^n+a_1x^{n-1}+\cdots+a_n=0$, with coefficients in F, are such that $p(x)=0$ has a Galois group relative to F which is representable as a subgroup G of the symmetric group S_n on n symbols, and which has the usual well-known properties. The method proposed by the author for determining G is to construct for the given equation, and for any given subgroup $\Gamma\subset S_n$, of order h, an "induced equation" of degree $k=n!/h$ [misprinted in the paper], with coefficients in F, such that $G\not\subseteq\Gamma$ if it has no roots in F, while $G\subseteq\Gamma$ if it has at least one simple root in F. He proves that such an induced equation exists, and, in fact, gives (theorems 1 and 2) general constructions for one. The more complicated second construction (theorem 2) need only be employed if the first happens to give an equation which has one or more multiple roots in F but no simple ones. In practice, simpler constructions than the general ones may yield a set of induced equations for different Γ's sufficing for the complete determination of G. The author illustrates this in the quartic case where, for example, the resolvent cubic is an induced equation for a subgroup of S_4 of order 8. *R. Hull.*

Murnaghan, Francis D. Schwarz' inequality and Lorentz spaces. Proc. Nat. Acad. Sci. U.S.A. **36**, 673–676 (1950).

Denoting $n\times 1$ matrices with real or complex elements by x, y and their conjugate transposes by x^*, y^*, the author gives a direct proof of Schwarz' inequality $(x^*x)(y^*y)\geqq(x^*y)(y^*x)$ with the case of linear dependence included. The main purpose of the paper is to give analogous inequalities in a Lorentz space for which the scalar product is x^*Fy, F being the $n\times n$ diagonal matrix with elements $(1, 1, \cdots, 1, -1)$. An $n\times 1$ matrix x is time-like, null, or space-like according as x^*Fx is negative, zero, or positive, respectively. An $n\times 2$ matrix X is space-like if every nontrivial linear combination of its two $n\times 1$ columns is space-like; then the 2×2 Hermitian matrix X^*FX is positive-definite. The analog of Schwarz' inequality is contained in the following statements, in which $(x^*Fx)(y^*Fy)-|x^*Fy|^2$ is denoted by D: (i) $D<0$ if, and only if, there exists a time-like linear combination of x and y, supposed linearly independent; (ii) if $D=0$, there exists a null nontrivial linear combination of x and y, while there does not exist a time-like linear combination; (iii) $D>0$ if, and only if, the $n\times 2$ matrix X, whose column matrices x and y are linearly independent, is space-like. The results are extended to the case of p $n\times 1$ matrices where $3\leqq p\leqq n$. Consideration is also given to spaces with more than one time direction; for example, a space in which the scalar product involves the $n\times n$ diagonal matrix with elements $(1, 1, \cdots, 1, -1, -1)$. *J. L. Synge.*

Yaglom, I. M. Quadratic and skew-symmetric bilinear forms in a real symplectic space. Trudy Sem. Vektor. Tenzor. Analizu. **8**, 364–381 (1950). (Russian)

The author gives a complete classification of symmetric and skew-symmetric bilinear forms on a real symplectic space, i.e., a linear space over the real field with a distinguished nonsingular skew-symmetric bilinear (ground) form. Two forms are considered equivalent if there is a symplectic transformation (i.e. a linear transformation on the space leaving invariant the ground form) taking one form into the other. The problem of classifying forms within this type of equivalence is essentially the same as that of classifying pairs of bilinear forms, of which one is skew-symmetric and nonsingular and the other either symmetric or skew-symmetric. The analogous problem for complex spaces was solved by Weierstrass and Kronecker, and there is no special difficulty in extending their approach to the real case when the latter form is skew-symmetric, but when it is symmetric certain invariants appear which are absent in the complex (or more generally, algebraically closed) case. The requirement that the ground form be nonsingular can be weakened by appealing to a result of Kronecker on pairs of forms. *I. E. Segal* (Chicago, Ill.).

Abstract Algebra

Mitrinovitch, Dragoslav S. Sur une propriété des opérations max et min. C. R. Acad. Sci. Paris **232**, 286–287 (1951).

The following generalization of the self-dual condition for a distributive lattice is a rather easy consequence of it: The union of all cross-cuts, k at a time, of n elements of a distributive lattice is equal to the cross-cut of all unions, $n-k+1$ at a time, of the same elements. Restricting attention entirely to chains of real numbers (union = max, etc.), the author proves this for the special, self-dual, case $k=\frac{1}{2}(n+1)$.

R. Church (Annapolis, Md.).

Mitrinovitch, Dragoslav S. Sur les operations max **et** min. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 4, 10 pp. (1950). (Serbo-Croatian. French summary)

This is a more detailed presentation of the material in the paper reviewed above. In addition the author observes that his result also holds for subsets of a set. The concluding remark involves a slight confusion between his result and the modular law. *R. Church* (Annapolis, Md.).

Mitrinovitch, Dragoslav S. Sur certaines relations de l'algèbre des ensembles. C. R. Acad. Sci. Paris **232**, 917–918 (1951).

Let the A_i be n elements in a distributive lattice and let $Q_j^{(k)}=\bigvee^{(k)} A_i$, $P_j^{(k)}=\bigwedge^{(k)} A_i$, $j=1, 2, \cdots, \binom{n}{k}$, be the unions and cross-cuts of k of these elements. Then the author proves that $\bigvee_j P_j^{(k)} \geqq \bigwedge_j Q_j^{(k)}$ when $1\leqq k\leqq \frac{1}{2}(n+1)$ and the reverse inclusion holds for the greater values of k; hence for n odd and $k=\frac{1}{2}(n+1)$, $\bigvee_j P_j^{(k)}=\bigwedge_j Q_j^{(k)}$ generalizing the well-known identity

$$(A_1\cap A_2)\cup(A_1\cap A_3)\cup(A_2\cap A_3) =(A_1\cup A_2)\cap(A_1\cup A_3)\cap(A_2\cup A_3).$$

O. Ore (New Haven, Conn.).

Livšic, A. H. On the Jordan-Hölder theorem in structures. Amer. Math. Soc. Translation no. **44**, 15 pp. (1951).

Translated from Mat. Sbornik N.S. **24(66)**, 227–235 (1949); these Rev. **10**, 674.

Bernstein, B. A. A dual-symmetric definition of Boolean algebra free from postulated special elements. Scripta Math. **16**, 157–160 (1950).

The author shows that the postulates (A) $ab=ba$, (B) $a(b\vee c)=ab\vee ac$, and (C) $a(b'\vee b)=a$, together with their duals, closure properties with respect to the operations, and the existence of at least two elements, are sufficient for Boolean algebra. (Here "$\vee$" indicates the Boolean (non-exclusive) sum, and a' is the complement of a. The postulates are shown to be independent. *H. B. Curry.*

Sikorski, Roman. On an analogy between measures and homomorphisms. Ann. Soc. Polon. Math. **23**, 1–20 (1950).

The author's program in this paper is to generalize to Boolean homomorphisms the statements obtained by specializing known measure theoretic extension theorems to two-valued measures. A typical result is that every σ-homomorphism from a subalgebra A_0 of a σ-algebra A into a σ-algebra B of sets can be extended to the least σ-algebra containing A_0. (The assumption that B is a σ-algebra of sets is essential.) *P. R. Halmos* (Montevideo).

Sikorski, Roman. Cartesian products of Boolean algebras. Fund. Math. **37**, 25–54 (1950).

The author presents a detailed study of the concept of Cartesian product (perhaps better called tensor product) of Boolean algebras. The principal difficulty arises in defining the tensor product of a family of σ-algebras so as to obtain a σ-algebra suitably related to the given factors. Two obvious conditions that the product might be required to satisfy are (1) that the isomorphism class of the product be uniquely determined by the isomorphism classes of the factors, and (2) that prescribed homomorphisms (or measures) on the factors be extendable to the product. The author points out that none of the obvious definitions of product satisfies both conditions and he proceeds, therefore, to study various extremal (i.e. maximal or minimal) products, and to show that they usually satisfy one of the two conditions. The technique involves, as is to be expected, heavy use of Stone's theory of topological representations.
P. R. Halmos (Montevideo).

Tamari, Dov. Représentations isomorphes par des systèmes de relations. Systèmes associatifs. C. R. Acad. Sci. Paris **232**, 1332–1334 (1951).

Continuing his previous work [e.g. same C. R. **229**, 1291–1293 (1949); these Rev. **11**, 327], the author gives examples of representations of various algebraic systems by families of relations. These include nonassociative systems, whereas Riguet [ibid. **231**, 936–937 (1950); these Rev. **12**, 472] assumed associativity. The adjunction of a unit, and the equivalence of associativity with some other conditions, are then considered. *P. M. Whitman* (Silver Spring, Md.).

***Pickert, Günter. Einführung in die höhere Algebra.** Vandenhoeck & Ruprecht, Göttingen, 1951. 298 pp. 12.80 DM; bound, 14.80 DM.

After the necessary preliminary notation and ideas have been presented, this book introduces the concept of an algebraic structure and develops a few fundamental properties from this general point of view. Then the structures are specialized in turn to groups, rings, fields, vector spaces, linear algebras, and lattices. Finally, the author presents in more detail the theory of integral domains, groups, fields, ordered fields, and valuations. The extent of the treatment is roughly the same as that of the first volume of van der Waerden [Moderne Algebra, 3d ed., Springer, Berlin, 1950; these Rev. **12**, 236], although with minor variations, and the spirit seems to be that of van der Waerden considerably influenced by Bourbaki. The excellence of the exposition and the presence of numerous lists of exercises should make the book suitable as a text for students who read German.
N. H. McCoy (Northampton, Mass.).

Pickert, Günter. Bemerkungen zum Homomorphiebegriff. Math. Z. **53**, 375–386 (1950).

The author discusses homomorphisms of systems with operations of a very general type. The main results are an analogue of Birkhoff's subdirect representation and extensions of the concept of the kernel for algebraic systems and their homomorphisms. In an added note it is pointed out that this theory is closely related to that of Shoda [Osaka Math. J. **1**, 182–225 (1949); these Rev. **11**, 308].
O. Ore (New Haven, Conn.).

Dörge, Karl. Bemerkung über Elimination in beliebigen Mengen mit Operationen. Math. Nachr. **4**, 282–297 (1951).

The author proves that the following theorem carries over to the realm of abstract algebras [cf. G. Birkhoff, Proc. Cambridge Philos. Soc. **31**, 433–454 (1935)]: If $\mathfrak{M}$ is a field, $\{\varphi_\alpha, \psi_\alpha\}$ a collection of polynomials in an arbitrary number of indeterminates over $\mathfrak{M}$, then the equations $\varphi_\alpha=\psi_\alpha$ have a common solution in some extension field of $\mathfrak{M}$ if and only if the polynomial ideal generated by the $\varphi_\alpha-\psi_\alpha$ contains no two elements of $\mathfrak{M}$; this happens exactly when each finite set of these equations has a solution in some algebra over $\mathfrak{M}$. The field $\mathfrak{M}$ is to be any abstract algebra with operations of finite index; "extension field" is to be replaced by "abstract algebra containing $\mathfrak{M}$ with no proper homomorphisms that are isomorphisms on $\mathfrak{M}$"; and for "ideal generated by the

$\varphi_\alpha - \psi_\alpha$ contains" read "homomorphic equivalence relation generated by the relations $\varphi_\alpha \sim \psi_\alpha$ identifies." The author notes generalizations of some individual steps to algebras with operations of infinite index. Most of the paper is devoted to definitions of the concepts involved.

D. Zelinsky (Evanston, Ill.).

Snapper, E. Completely primary rings. III. Imbedding and isomorphism theorems. Ann. of Math. (2) **53**, 207–234 (1951).

This is a continuation of part II [same vol., 125–142 (1951); these Rev. **12**, 387]. First, the author develops the necessary field theory. If K is a fixed extension field of the fixed field L of characteristic p, K is called a separable extension if K preserves p-independence. A separating set is a nonempty algebraically independent subset T of K such that K is a separable extension of $L(T)$. If K is a separable extension and contains no separating sets, K is called a relatively perfect extension. The field K is a separable extension if and only if K can be obtained by a purely transcendental extension followed by a relatively perfect extension. A number of results about relatively perfect extensions are established. If the ring S is an extension of the ring R (ring always means completely primary ring), then the field $\bar{S}$ is an extension of the field $\bar{R}$. The main problem of the imbedding theory is the following. If $\bar{T}$ is a field such that $\bar{R}\subseteq\bar{T}\subseteq\bar{S}$, when will S contain a principal extension ring of R whose residue class field is $\bar{T}$? It is shown that the imbedding problem always has a solution if $\bar{T}$ has a separating transcendence basis over $\bar{R}$. We now also consider all field extensions of characteristic zero to be separable field extensions. If the radical of S is nilpotent, the imbedding problem has a solution if $\bar{T}$ is any separable field extension of $\bar{R}$.

If $R\subset S$, and $\mathfrak{a}$ is an ideal in R, the extension of $\mathfrak{a}$ in S is the ideal in S generated by the elements of $\mathfrak{a}$. If $\mathfrak{b}$ is an ideal in S, the contraction of $\mathfrak{b}$ in R is $\mathfrak{b}\cap R$. A principal ring extension $R\subset S$ is called a canonical extension if every ideal in R coincides with the contraction of its extension. Canonical ring extensions are studied in detail in preparation for the ring isomorphism theory. Now assume that $R\subset S$ is a fixed ring extension, $R'\subset S'$ another fixed ring extension, and that there exists a fixed isomorphism J_0 from R onto R'. The isomorphism from $\bar{R}$ onto $\bar{R}'$ induced by J_0 may be denoted by $\bar{J}_0$. Let $\bar{J}$ denote a fixed isomorphism of $\bar{S}$ onto $\bar{S}'$ which is an extension of $\bar{J}_0$. The main problem of the isomorphism theory is to determine when J_0 can be extended to an isomorphism J from S onto S' such that J induces the given isomorphism $\bar{J}$ from $\bar{S}$ onto $\bar{S}'$. The author shows that the problem always has a solution if $R\subset S$ and $R'\subset S'$ are canonical separable (i.e., the corresponding field extensions are separable) extensions with nilpotent radicals. A number of other related results are also obtained.

N. H. McCoy (Northampton, Mass.).

Peremans, W. The radical of a ring. Math. Centrum Amsterdam. Rapport **ZW-1950-013**, 11 pp. (1950). (Dutch. English summary)

An expository paper giving various definitions of the radical of a ring, and discussing the relationships between them. *N. H. McCoy* (Northampton, Mass.).

Foster, Alfred L. p-rings and their Boolean-vector representation. Acta Math. **84**, 231–261 (1951).

Ce que fait l'auteur revient à ceci. Soit A un p-anneau, c'est-à-dire un anneau commutatif avec élément unité tel que $x^p=x$ et $px=0$ dans A (p nombre premier); d'après un théorème de McCoy et Montgomery [Duke Math. J. **3**, 455–459 (1937)] A peut être considéré comme un sous-anneau d'un produit $F_p{}^E$, où E est un ensemble arbitraire et F_p le corps premier à p éléments. Autrement dit, chaque $x\varepsilon A$ est une fonction $t\rightarrow x(t)$ définie dans E et à valeurs dans F_p. Pour chaque élément $r\neq 0$ de F_p (on peut prendre $r=1, 2, \cdots, p-1$) soit y_r la fonction caractéristique de l'ensemble des $t\varepsilon E$ où $x(t)=r$; on a donc $x=\sum_{r=1}^{p-1} ry_r$; grâce au fait que le groupe multiplicatif $F_p{}^*$ est cyclique, les éléments y_r sont inversement des combinaisons linéaires de $x, x^2, \cdots, x^{p-1}$ à coefficients dans F_p, donc appartiennent à l'anneau A; l'auteur donne explicitement les expressions des y_r en fonction des x^r, et associe à tout $x\varepsilon A$ sa "décomposition normale" $(y_1, \cdots, y_{p-1})$; puis il traduit les opérations sur les éléments de A en opérations sur les "décompositions normales" associées. *J. Dieudonné* (Nancy).

Kaplansky, Irving. Locally compact rings. II. Amer. J. Math. **73**, 20–24 (1951).

Continuing an earlier paper [same J. **70**, 447–459 (1948); these Rev. **9**, 562], the radical of a locally compact ring is studied in more detail. It is shown that the radical is closed. It is in fact the intersection of the closed regular right (or left) ideals. Next two decomposition theorems are proved using Braconnier's local direct sum [J. Math. Pures Appl. (9) **27**, 1–85 (1948); these Rev. **10**, 11]. (1) If the idempotents of the ring are in the center then the ring is a local direct sum of Q-rings [see the author, same J. **69**, 153–183 (1947); these Rev. **8**, 434]. (2) In generalization of results for rings where every element satisfies $a^{n(a)}=a$, $n(a)>1$, strongly regular rings [see Arens and the author, Trans. Amer. Math. Soc. **63**, 457–481 (1948); these Rev. **10**, 7] are shown to be decomposable into a discrete ring, a direct sum of a finite number of nondiscrete division rings and a local direct sum of discrete strongly regular rings relative to finite subfields. *O. Todd-Taussky* (Washington, D. C.).

Hua, Loo-Keng. On the multiplicative group of a field. Acad. Sinica Science Record **3**, 1–6 (1950). (English. Chinese summary)

Let K be a noncommutative division ring, K_0 the multiplicative group of nonzero elements. The author proves that K_0 is not solvable, that is, its derived series cannot reach the identity in a finite number of steps. Use is made of the results and methods of an earlier paper [Proc. Nat. Acad. Sci. U.S.A. **35**, 533–537 (1949); these Rev. **11**, 155].

I. Kaplansky (Chicago, Ill.).

Grell, Heinrich. Modulgruppen und -inversionen bei primären Integritätsbereichen. Math. Nachr. **4**, 392–407 (1951).

Let R be a primary integral domain with restricted minimum condition, K its quotient field. For ideals I, J (which may be fractional), I/J is the set of $c\varepsilon K$ with $cJ\subset I$. The ideal I is exact if $I/I=R$, R-invertible if $R/(R/I)=I$. Invertible ideals coincide with principal ideals and also with R-invertible exact ones. Let R be called unforked (unvergabelt) if there is exactly one ideal immediately larger than R. Various properties of unforked rings are studied; for example, this property is equivalent to the R-invertibility of every finitely generated ideal. The remainder of the paper is devoted to an abstract inversion operation $I\rightarrow I'$ which satisfies the axioms $I''=I$, $(IJ)'=I'/J=J'/I$. Unforkedness of R is equivalent to the existence of such an inversion, together with the property that the exact ideals form a group. In rings of algebraic integers, Dedekind introduced

such an inversion by taking complementary bases relative to the trace; this method rests on separability. The author promises to show in a later paper that his approach works without separability. *I. Kaplansky* (Chicago, Ill.).

Rédei, L. Ein Satz über quadratische Formen. Math. Ann. **122**, 340–342 (1950).

Let K be a formally real field and R a subring of K which is a principal ideal ring and which contains the unit element 1 of K. Let $f(x)=\sum a_{ij}x_ix_j$ $(i, j=1, \cdots, n)$ be a positive definite form with coefficients in K, and $n\geqq 2$. If $c_1, \cdots, c_n$ are elements of K, set $f(c, x)=\sum a_{ij}c_ix_j$. If the c_i are not all zero, then $\bar{f}(x)=f(c)f(x)-f^2(c, x)$ is equivalent to a positive definite quadratic form in $n-1$ indeterminates. If the c_i are relatively prime elements of R, there exists a homogeneous linear transformation with coefficients in R and of determinant 1, which transforms $\bar{f}(x)$ into a form containing $n-1$ indeterminates. If D is the determinant of $f(x)$, the determinant of the transformed form differs from $Df^{n-2}(c)$ by at most a factor which is a unit of R. *N. H. McCoy.*

Kawada, Yukiyosi, and Iwahori, Nagayosi. On the structure and representations of Clifford algebras. J. Math. Soc. Japan **2**, 34–43 (1950).

This is another of the many papers on the structure and representations of the associative algebras $F[u_1, \cdots, u_n]$ where $u_i^2=\alpha_i\neq 0$ in F and $u_iu_j+u_ju_i=0$ for $i\neq j$. These authors consider the case where F is any field of characteristic not two and each $\alpha_i=\pm 1$. The results are trivial in view of the procedure outlined by the reviewer [cf. the review of H. C. Lee, Ann. of Math. (2) **49**, 760–773 (1948); these Rev. **10**, 180]. *A. A. Albert* (Chicago, Ill.).

Iwahori, Nagayoshi, and Satake, Ichirô. On Cartan subalgebras of a Lie algebra. Kōdai Math. Sem. Rep. **1950**, 57–60 (1950).

If L is a Lie algebra over the field of complex numbers, then any two Cartan subalgebras of L may be transformed into each other by some operation of the adjoint group of L [Chevalley, Amer. J. Math. **63**, 785–793 (1941); these Rev. **4**, 2]. The authors observe that this is not the case any more if the basic field is the field of real numbers. However, they prove that, if L is solvable, then the theorem is true for any basic field of characteristic 0, the adjoint group being replaced by the group generated by all operations of the form exp (ad X), for those X whose adjoints are nilpotent. From this, they deduce that, R being the radical of L, and the basic field being that of real numbers, if any two Cartan subalgebras of the semisimple L/R may be transformed into each other by an operation of the adjoint group of L/R, then any two Cartan subalgebras of L may be transformed into each other by an operation of the adjoint group of L. The proof is based on the theory of algebraic Lie algebras. *C. Chevalley* (New York, N. Y.).

Karpelevič, F. I. On nonsemisimple maximal subalgebras of semisimple Lie algebras. Doklady Akad. Nauk SSSR (N.S.) **76**, 775–778 (1951). (Russian)

Let G be a semisimple Lie algebra, Σ and Π systems of roots and simple roots respectively. To a maximal nonsemisimple subalgebra G_1 there is attached a subsystem Σ_1 of Σ. The author first shows that $\Sigma_1\cup(-\Sigma_1)=\Sigma$. In the remaining investigation, the hypothesis of maximality is replaced by this weaker condition. After an inner automorphism, Σ_1 can be described as the set of all roots having nonnegative coefficients on a certain subset Π_+ of Π. The case of maximality is that where Π has just one element. *I. Kaplansky* (Chicago, Ill.).

Dynkin, E. B. Automorphisms of semi-simple Lie algebras. Doklady Akad. Nauk SSSR (N.S.) **76**, 629–632 (1951). (Russian)

Let α_i be a system of simple roots of a semi-simple Lie algebra G and $e(\alpha_i)$ corresponding root vectors. Let $\alpha_i\rightarrow\alpha_i'$ be an isometry of $\{\alpha_i\}$ relative to the Cartan metric. It is known that there exists a unique automorphism f of G satisfying $f(\alpha_i)=\alpha_i'$, $f[e(\alpha_i)]=e(\alpha_i')$. Let B be the group of these automorphisms, A the group of all automorphisms of G, A_0 the group of inner automorphisms. Then A is the semi-direct product of B and A_0. The proof proceeds by a succession of normalizations of a proposed automorphism of G, and depends heavily on structure theory. An application is made to representation theory; it is shown that every irreducible representation of the algebras B_n, C_n, D_{2k}, G_2, F_4, E_7, and E_8 is self-contragredient, while an irreducible representation of A_n, D_{2k+1}, E_6 is self-contragredient if and only if the highest weight is invariant under B. *I. Kaplansky* (Chicago, Ill.).

Malcev, A. I. Commutative subalgebras of semi-simple Lie algebras. Amer. Math. Soc. Translation no. **40**, 15 pp. (1951).

Translated from Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **9**, 291–300 (1945); these Rev. **7**, 362.

Theory of Groups

Bruck, R. H. An extension theory for a certain class of loops. Bull. Amer. Math. Soc. **57**, 11–26 (1951).

Das Erweiterungsproblem der Gruppentheorie, zu gegebenen Gruppen K, M alle Gruppen E aufzusuchen, so dass K in E Normalteiler und $E/K=M$ ist, wird auf loops ausgedehnt und gibt so Anlass zu Untersuchungen, die in engem Zusammenhang mit der Homologietheorie von Eilenberg und MacLane stehen [siehe z.B. Eilenberg und MacLane, Ann. of Math. (2) **48**, 51–78, 326–341 (1947); Duke Math. J. **14**, 435–463 (1947); Eilenberg, Bull. Amer. Math. Soc. **55**, 3–37 (1949); diese Rev. **8**, 367; **9**, 7, 132; **11**, 8]. Die Arbeit enthält viele Einzelresultate, in denen sich zum Teil bekannte Sätze wiederfinden. Ist G eine abelsche Gruppe, M ein loop, so bedeutet das Symbol (G, M), dass für x, y aus M und g, g' aus G ein eindeutiges Produkt gx aus G erklärt ist, wobei $g\cdot 1=g$, $(gg')x=(gx)(g'x)$, $(gx)y=g(xy)$. Sei $A(E)$ der Assoziator des loop, d.h. die Menge der Elemente aus E, die der Bedingung $x(yz)=(xy)z$ genügen, wenn mindestens eines der drei Elemente in A liegt. Eine (E, θ)-Erweiterung (extension) von (G, M) wird definiert durch ein loop E, eine Homomorphie θ von E auf M, mit dem Kern K aus $A(E)$, dem Kernzentrum G und der Eigenschaft $ge=e(gx)$ für g aus G, e aus E, $x=e\theta$ aus M. Ist $1\theta^{-1}=G$, so heisst die Erweiterung zentral. Unter den (E, θ)-Erweiterungen werden verschiedene Typen unterschieden hinsichtlich der Beklammerungsregeln, die in M gelten bzw. die E aus M übernimmt. Besonderes Interesse bieten die Fälle, in denen M eine Gruppe ist oder ein loop mit dem abgeschwächten Assoziativgesetz $xy\cdot zx=x(yz\cdot x)$ (als Moufang-loop bezeichnet). Sei W_n ein Wort in einem freien loop aus n freien Erzeugenden [siehe G. E. Bates, Amer. J. Math. **69**, 499–550 (1947); diese Rev. **9**, 8], so

heisst $W_n(e_1, \cdots, e_n)$ rein nicht assoziativ (r.n.a.), wenn W_n in jeder Gruppe dem Einselement gleich wird. Wenn aus $W_n(e_1, \cdots, e_n)\theta = 1$ folgt $W_n(e_1, \cdots, e_n) = 1$, so heisst die (E, θ)-Erweiterung von (G, M) stark gruppenartig (strongly grouplike, s.g.). Ist dagegen C eine Menge von r.n.a. Worten, so dass für jedes W_n aus C für alle x aus M die Bedingung $W_n(x_1, \cdots, x_n) = 1$ gilt, dann heisst eine (E, θ)-Erweiterung von (G, M) "C-artig", wenn für alle W_n aus C und alle e_i aus E ebenfalls gilt $W_n(e_1, \cdots, e_n) = 1$. Sei z.B. C definiert als die Gesamtheit der Elemente B_3, die die Gleichungen erfüllen $x_1x_2 \cdot x_3x_1 = x_1(x_2x_3 \cdot x_1) \cdot B_3(x_1, x_2, x_3)$, und sei M ein Moufang-loop, so wird die zugehörige C-artige Erweiterung als eine Moufang-Erweiterung bezeichnet. In einem freien loop fallen die C-artigen Erweiterungen mit den s.g. Erweiterungen zusammen. Für Moufang-loops gelten insbesondere die Strukturaussagen, die Ref. früher abgeleitet hatte [Math. Ann. **110**, 416–430 (1934)], und weitere Sätze, die sich topologisch als Aussagen über 2-Koketten bzw. 3-Koränder aussprechen lassen. Das Studium der (E, θ)-Erweiterungen enthält 3 Hauptgesichtspunkte: (1) Die Theorie der Äquivalenz zweier Erweiterungen (E, θ_1), (E, θ_2), in die Verf. noch den Begriff der Inversen auch für nicht-zentrale Erweiterungen einführt. Die Gesamtheit der (E, θ)-Erweiterungen von (G, M) erweist sich bei geeigneter Verknüpfung $\otimes$ als eine kommutative Halbgruppe S mit Einselement, wenn die Äquivalenz als Gleichheitsbeziehung benutzt wird. (2) Die r.n.a. Worte $W_n(e_1, \cdots, e_n)$ haben die fundamentale Eigenschaft, nur eine von den Bildern $x_i = e_i\theta$ abhängende Funktion $F(W_n, E; x_1, \cdots, x_n)$ zu sein. An jeder Einsstelle $x_1, \cdots, x_n$ von W_n bilden die F bei festem W_n über die E eine multiplikative abelsche Gruppe. Die Funktionen F verallgemeinern die Eilenberg-MacLaneschen Kozyklen; ist A_3 erklärt durch $(x_1 \cdot x_2)x_3 = x_1(x_2x_3) \cdot A_3(x_1, x_2, x_3)$, so erweist sich $F(A_3, E)$ als ein 3-Kozyklus. (3) Ist M eine Gruppe, so lassen sich in der üblichen Weise die n-Koketten als eindeutige Funktionen f_n erklären, die für Argumente $x_1, \cdots, x_n$ aus M Werte aus G annehmen, und dazu die Koränder δf_n. Ist $f(x, y)$ eine normale 2-Kokette zu (G, M), so wird (G, M, f) als die zentrale (E, θ)-Erweiterung von (G, M) definiert, für die die Elemente von E die Paare (x, g) sind, x aus M, g aus G, mit der Gleichheitsdefinition $(x, g) = (y, g')$ für $x = y$, $g = g'$; ferner sei $(x, g)(y, g') = (xy, f(x, y) \cdot (gy)g')$, und $(1, g) = g$. Jede zentrale Erweiterung von (G, M) ist dann äquivalent zu mindestens einer Erweiterung (G, M, f). Die Äquivalenz zweier solcher Erweiterungen und ihre $\otimes$-Verknüpfung drücken sich dann in einfacher Weise durch die f und δf aus. Die Erweiterung (G, M, f) ist genau dann gruppenartig, d.h. für jede Untergruppe H von M ist $H\theta^{-1}$ eine Untergruppe von E, wenn $(\delta f)(x, y, z) = 1$ für alle x, y, z, die eine Untergruppe von M erzeugen. Die Bedingung dafür, dass die zentrale Erweiterung (G, M, f) eine Moufang-Erweiterung ist, drückt sich in einer Funktionalgleichung für $f(x, y)$ aus. Diese Erweiterungen bilden eine Untergruppe der Gruppe aller zentralen Erweiterungen. Für Moufang-loops endlicher Ordnung m und für eine zentrale Moufang-Erweiterung (E, θ) erhält Verf. eine Reihe von Sätzen über die loops E^a (a ein Element aus $A(M)$), E^{mn} (n = kleinstes gemeinsames Vielfache der Ordnungen der Elemente von M); diese Sätze verallgemeinern ein Resultat von M. Hall über zentrale assoziative Erweiterungen von (G, M), falls M eine Gruppe ist [Ann. of Math. (2) **39**, 220–234 (1938)].

Für die Topologie interessieren noch gewisse abelsche Gruppen $\mathfrak{Z}$, $\mathfrak{B}$, $\mathfrak{H} = \mathfrak{Z}/\mathfrak{B}$ von Erweiterungen; sie sind erklärt mit Hilfe des Begriffes Nucleus der kommutativen Semigruppe S, das ist die Gesamtheit der Elemente von S, die den Kern einer homomorphen Abbildung von S auf eine Gruppe bilden. Ist $N_{s.g.}$ die Menge der stark gruppenartigen Erweiterungen, $N'_{s.g.}$ der Durchschnitt der zentralen Erweiterungen mit $N_{s.g.}$, so ist $\mathfrak{Z}_{s.g.} = S/N_{s.g.}$, $\mathfrak{B}_{s.g.} = (S' \otimes N_{s.g.})/N_{s.g.}$, $\mathfrak{H}_{s.g.} = \mathfrak{Z}_{s.g.}/\mathfrak{B}_{s.g.}$. Analog sind $\mathfrak{Z}_C$, $\mathfrak{B}_C$, $\mathfrak{H}_C$ für C-artige Erweiterungen definiert. Diese Gruppen stehen für $n = 3$ in Beziehung zu den Gruppen $\mathfrak{Z}_n$ der Koketten, $\mathfrak{B}_n$ der Koränder und den Kohomologiegruppen $\mathfrak{H}_n = \mathfrak{Z}_n/\mathfrak{B}_n$; wenn M eine Gruppe ist, induziert der Homomorphismus von (E, θ) auf $F(A_3, E)$ einen Isomorphismus von $\mathfrak{H}$ auf $\mathfrak{H}_3$. Ist M eine Gruppe und die C-Erweiterung eine Moufang-Erweiterung, so erzeugt der Homomorphismus von (E, θ) auf $F(B_3, E)$ einen Isomorphismus von $\mathfrak{H}_C$ auf $\mathfrak{H}_{3\rho}$, wo $\mathfrak{H}_{3\rho}$ das homomorphe Bild von $\mathfrak{H}_3$ ist bei der Abbildung $(f_3\rho)(x, y, z) = f_3(x, y, zx) \cdot f_3(y, z, x)^{-1}$. Diese Sätze stehen in Analogie zu bekannten Sätzen von Baer [Trans. Amer. Math. Soc. **58**, 295–347, 348–389, 390–419 (1945); diese Rev. **7**, 371, 372] und von Eilenberg und MacLane [loc. cit.] über zentrale Gruppenerweiterungen.

R. *Moufang* (Frankfurt am Main).

Ballieu, Robert. Sur les groupes de parties d'un demi-groupe. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 139–147 (1950).

The subject matter of this paper is the semigroup $P(G)$ consisting of all subsets of a semigroup G, and more particularly its subgroups. The following two results are typical of those obtained. (i) If H is a subgroup of $P(G)$ and the unit element of H is a subgroup of G then the union of all the subsets of G which are the elements of H is also a group and the group which is the unit element is a normal divisor of this group. (ii) If G is itself a group and K_0 is a subset of G, then the sets K_0x, x in G, form a group if and only if there exists u in G such that $KxK = Kx$ for all x in G, where $K = K_0u$. Further, if K^{-1} is the set of inverses of elements of K, then KK^{-1} is a normal divisor of K.

D. Rees.

Suzuki, Michio. On the lattice of subgroups of finite groups. Trans. Amer. Math. Soc. **70**, 345–371 (1951).

The author adds many new theorems to the existing knowledge of the extent to which a group G is determined by the lattice $L(G)$ of its subgroups. The number of types of groups lattice-isomorphic to a given finite group G is proved finite if $L(G)$ has no chain as direct component. The structure of groups lattice-isomorphic to a p-group is determined, followed by theorems on groups with nonindex-preserving lattice-isomorphisms. If ϕ is a lattice-isomorphism between G and G', then there exists a normal subgroup N of G with $N' = \phi(N)$ normal in G', N' having the same order as N; if H is a direct factor of G/N or G'/N' then H is cyclic or $L(H)$ is an irreducible complemented modular lattice. A lattice-isomorphism of a perfect group is always index-preserving, and preserves normality and the center; for any group it preserves the radical (maximal solvable normal subgroup). If $G = G_1 \times G_2$ and G_1 or G_2 is perfect (or if the radical of G is the unit) and G is lattice-isomorphic to H under ϕ, then $H = \phi(G_1) \times \phi(G_2)$. If the center of G is the unit, then any lattice-automorphism of G is induced by at most one element-automorphism of G. If G is simple, then G is element-isomorphic to H if and only if $G \times G$ and $H \times H$ are lattice-isomorphic. The structure of nilpotent groups, and finite solvable groups, with dual (lattice anti-isomorphic) groups is determined.

P. M. Whitman.

Suzuki, Michio. On the L-homomorphisms of finite groups. Trans. Amer. Math. Soc. **70**, 372–386 (1951).

The author obtains independently a characterization of groups lattice-homomorphic to a cyclic group, already studied by Zappa [Rend. Sem. Mat. Univ. Padova **18**, 140–162 (1949); these Rev. **11**, 156], and finds a similar characterization if the image group is assumed nilpotent instead. Some properties of groups G admitting proper lattice-homomorphisms are found; for instance, if G is a p-group then G is either cyclic or a generalized quaternion group. Any lattice-homomorphic image of a perfect or solvable group is itself perfect or solvable respectively. The neutral elements of any lattice are characterized.

P. M. Whitman (Silver Spring, Md.).

Specker, Ernst. Additive Gruppen von Folgen ganzer Zahlen. Portugaliae Math. **9**, 131–140 (1950).

Let F be the additive group of sequences $\{a_n\}$ of integers. A growth-type is any subset ϕ of the totality χ of increasing sequences $\{p_n\}$ of natural numbers such that (1) $\{p_n\}\epsilon\phi$, $\{q_n\}\epsilon\chi$, $q_n\leqq p_n$ imply $\{q_n\}\epsilon\phi$; (2) $\{p_n\}\epsilon\phi$, $\{q_n\}\epsilon\phi$ imply $\{p_n+q_n\}\epsilon\phi$. To each growth type ϕ is associated a subgroup F_ϕ of F: $\{a_n\}\epsilon F_\phi$ if $\{\max_{i\leqq n}(1, |a_i|)\}\epsilon\phi$. The growth-type consisting of all bounded sequences in χ is denoted by η. It is shown that the totality of growth-types has cardinal number $2^{\aleph}$; that $F_\phi\cong F_\psi$ only if $\phi=\psi$; that all subgroups of F of cardinal number $\aleph_0$ and all subgroups of F_η of cardinal number $\aleph_1$ are free Abelian; and that if $\phi\neq\eta$, F_ϕ has a non-free subgroup of cardinal number $\aleph_1$. *P. A. Smith.*

Hirsch, Kurt A. Eine kennzeichnende Eigenschaft nilpotenter Gruppen. Math. Nachr. **4**, 47–49 (1951).

The group G satisfying the ascending chain condition for its subgroups is nilpotent (i.e., possesses a (finite) central chain from 1 to G) if, and only if, every proper subgroup of G is different from its normalizer. *R. Baer.*

Smirnov, D. M. On the theory of locally nilpotent groups. Doklady Akad. Nauk SSSR (N.S.) **76**, 643–646 (1951). (Russian)

A subgroup H of a group G is called infra-invariant in G provided for every inner automorphism φ of G, either $\varphi(H)\subseteq H$ or $\varphi(H)\supseteq H$. The concepts of infra-invariant subgroup and normal subgroup coincide not only when G is periodic, but also when G is locally nilpotent. A group is Hamiltonian when each of its cyclic subgroups is infra-invariant. Let the group G have an upper central series $1=Z_0\subset Z_1\subset\cdots\subset Z_\gamma=G$ with $\gamma\leqq\omega$ such that the maximal periodic subgroup of each Z_n/Z_{n-1}, for $n\geqq 2$, satisfies the minimal condition for subgroups; if Z_1 is finite, so is G; if Z_1 is periodic, G need not be. If a torsion-free group has an upper central series with $\gamma\leqq\omega$ and if Z_1 contains no complete subgroup, then neither G nor any factor of the series contains any complete subgroup. In a locally nilpotent torsion-free group the normalizer of any servant subgroup is also servant. Other results pertain to Abelian subgroups of types A_3, A_4, A_5 as classified by Mal'cev [Doklady Akad. Nauk SSSR (N.S.) **67**, 23–25 (1949); these Rev. **11**, 78].

R. A. Good (College Park, Md.).

Plotkin, B. I. On the theory of locally nilpotent groups. Doklady Akad. Nauk SSSR (N.S.) **76**, 639–641 (1951). (Russian)

Let the group $\mathfrak{G}$ satisfy the normalizer condition, or equivalently, be special as defined by Šmidt [Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 363–375 (1940); these Rev. **2**, 214]. Then $\mathfrak{G}$ is locally nilpotent. Corollaries answer certain problems stated by Kuroš and Černikov [Uspehi Matem. Nauk (N.S.) **2**, no. 3(19), 18–59 (1947); these Rev. **10**, 677].

R. A. Good (College Park, Md.).

Muhammedžan, H. H. On groups with an ascending central series. Mat. Sbornik N.S. **28(70)**, 185–196 (1951). (Russian)

In this review, G is a group with [transfinite] ascending central series (a.c.s.) $\{Z_n\}$, $0\leqq n\leqq f$, $Z_0=1$, $Z_f=G$; i.e. Z_{n+1}/Z_n is the center of G/Z_n. Proofs of the following results are given. If H is a normal subgroup of G, such that $H\cap(Z_{r+1}-Z_r)$ is not empty, then $H\cap(Z_r-Z_s)$ is not empty ($r>s$). In particular, $H\cap(Z_1-Z_0)$ can never be empty. If moreover H is locally normal (every finite set of elements of H lies in a finite and normal subgroup of H) then $H\subset Z_\omega$; here ω is best possible. Let g ($g\epsilon G$) be expressible as a product of nth powers for each natural number n. Then g is permutable with every element of finite order in Z_ω [cf. Černikov, Rec. Math. [Mat. Sbornik] N.S. **18(60)** 397–422 (1946); these Rev. **8**, 311]. If all factors Z_{n+1}/Z_n are finite, then G is special, i.e., G satisfies the descending chain condition (d.c.c.) for subgroups. More generally, if all factors Z_{n+1}/Z_n satisfy d.c.c., then G/Z_1 has a.c.s., all factors of which are finite; in this case the length of the a.c.s. of G is not ω and is less than $\omega 2$. Conversely, a special group has a.c.s., all factors of which satisfy d.c.c. See the author's preliminary announcement [Doklady Akad. Nauk SSSR (N.S.) **65**, 269–272 (1949); these Rev. **10**, 590].

J. L. Brenner (Pullman, Wash.).

Pickert, Günter. Remaksche Zerlegungen für Gruppen mit Paarungen. Math. Z. **53**, 456–462 (1951).

An operator o in a group G produces an image $o(g)$ of a given element g in G and the result may be represented as a pair $(g, o(g))$ in the product group $G\times G$ or as a relation in G itself. The author therefore generalizes the operator concept to be a set of pairs (g, g') in $G\times G$. One can then introduce permissible subgroups under this pair operation and prove as an analogue to Remak's direct decomposition theorem for groups that when the ascending and descending chain conditions hold for the normal permissible subgroups there exist direct permissible decompositions into direct indecomposable components and any two such decompositions have isomorphic components. *O. Ore.*

Murnaghan, Francis D. The characters of the symmetric group. Proc. Nat. Acad. Sci. U.S.A. **37**, 55–58 (1951).

A method is described giving formulae for the characters of the symmetric groups of any degree in terms of the numbers and orders of the cycles. The differential coefficient of a partition $(\mu_1, \cdots, \mu_k)$ with respect to s_p is defined as

$$(\mu_1, \cdots, \mu_k)_p=(\mu_1-p, \mu_2, \cdots, \mu_k)+(\mu_1, \mu_2-p, \cdots, \mu_k)+\cdots+(\mu_1, \cdots, \mu_k-p).$$

In the expression on the right, if the parts are in nondescending order these are reduced to descending order by the usual convention $(\cdots, q, r, \cdots)=-(\cdots, r-1, q+1, \cdots)$. The characteristic of the class $(1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\cdots)$ of the symmetric group of order $m!$ corresponding to the partition $(m-\mu_1-\mu_2-\cdots-\mu_k, \mu_1, \mu_2, \cdots, \mu_k)$ is a polynomial in $\alpha_1, \alpha_2, \cdots$ which is denoted by $[\mu_1, \cdots, \mu_k]$. It can be expressed in such a manner that for all i it is linear in the binomial coefficients $1, \alpha_i, \binom{\alpha_i}{2}, \binom{\alpha_i}{3}, \cdots$. The integral of the expression with respect to α_i is defined as the expression obtained by replacing 1 by α_i, α_i by $\binom{\alpha_i}{2}$, etc., treating the other α_j as

constants. The terms in $[\mu_1, \cdots, \mu_k]$ which are independent of $\alpha_2, \alpha_3, \cdots$ may be obtained from Frobenius' formula for the degree of the representation corresponding to $(m-\mu_1-\cdots-\mu_k, \mu_1, \cdots, \mu_k)$, by putting $m=\alpha_1$. The terms involving, e.g., $\alpha_i\binom{\alpha_j}{2}$ but otherwise independent of $\alpha_2, \alpha_3, \cdots$ may be obtained by differentiating the partition $[\mu_1, \cdots, \mu_k]$ once with respect to s_i and twice with respect to s_j, taking the terms independent of $\alpha_2, \alpha_3, \cdots$, and then integrating once with respect to α_i and twice with respect to α_j. This procedure gives the required formula. Illustrative examples are given. *D. E. Littlewood* (Bangor).

Murnaghan, Francis D. On the analysis of representations of the linear group. Proc. Nat. Acad. Sci. U.S.A. **37**, 51–55 (1951).

The matrix representations of the full linear group correspond to the partitions (λ) of all integers. The matrix corresponding to (λ) is called an invariant matrix of the fundamental matrix of transformation A, and is denoted by $A^{[\lambda]}$. Its spur is called, in honor of I. Schur, an S-function. The invariant matrix of an invariant matrix, say $[A^{[\lambda]}]^{[\mu]}$ is equivalent to a direct sum of invariant matrices of the form $A^{[\nu]}$, and taking the spur one writes $\{\lambda\}\otimes\{\mu\}=\sum\{\nu\}$. The operation $\otimes$ which was originally called "new multiplication" of S-functions, and is so termed by the author, has more recently been termed the "plethysm" of S-functions. The expansion of such plethysms is of central importance in the theory of algebraic invariants. The S-function $\{\lambda\}$ is a symmetric function of the latent roots of the fundamental matrix A, and can thus be expressed in terms of the power sums $S_1, S_2, \cdots$. The derivative $\partial\{\lambda\}/\partial S_i$ is denoted by $\{\lambda\}_i$. A previous result obtained by the reviewer can thus be expressed as $[\{\lambda\}\otimes\{\mu\}]_1=[\{\lambda\}\otimes\{\mu\}_1]\{\lambda\}_1$. For low degrees this leads to a quick method of expanding the plethysm, but as the degrees increase ambiguities arise which render the method impractical. The author augments this result by further formulae which dispose of the arising ambiguities and thus render the method useful in these cases of higher degree. The formulae obtained may be written

$$\begin{aligned}
[\{\lambda\}\otimes\{\mu\}]_2&=[\{\lambda\}\otimes\{\mu\}_1]\{\lambda\}_2+[\{\lambda\}\otimes\{\mu\}_2][\{\lambda\}_1\otimes S_2];\\
[\{\lambda\}\otimes\{\mu\}]_3&=[\{\lambda\}\otimes\{\mu\}_1]\{\lambda\}_3+[\{\lambda\}\otimes\{\mu\}_3][\{\lambda\}_1\otimes S_3];\\
[\{\lambda\}\otimes\{\mu\}]_4&=[\{\lambda\}\otimes\{\mu\}_1]\{\lambda\}_4+[\{\lambda\}\otimes\{\mu\}_2][\{\lambda\}_2\otimes S_2]\\
&\qquad+[\{\lambda\}\otimes\{\mu\}_4][\{\lambda\}_1\otimes S_4].
\end{aligned}$$

The generalization is apparent. *D. E. Littlewood.*

Mautner, F. I. The structure of the regular representation of certain discrete groups. Duke Math. J. **17**, 437–441 (1950).

Démonstration du résultat suivant: Soit G un groupe discret dénombrable, soit G_0 le sous-groupe des $x\varepsilon G$ ayant un nombre fini de conjugués distincts, et supposons G/G_0 infini; alors, si l'on décompose la représentation régulière gauche de G relativement au centre de l'anneau qu'elle engendre, presque toutes les composantes obtenues sont de dimension infini (en d'autres termes, les composantes irréductibles de la "double représentation régulière" sont de classe (II_1) et non (I_n)). La démonstration consiste à former, pour chaque classe G_i modulo G_0, l'opérateur de projection E_i qui fait passer (dans L^2) d'une fonction $f(x)$ à la fonction $\theta_i(x)f(x)$, où $\theta_i=1$ sur G_i, $=0$ ailleurs, et à remarquer que ces E_i, qui sont décomposables, ont des composantes presque partout non nulles. Ce résultat implique que, dans tout groupe discret G dont la double représentation régulière se décompose en facteurs de dimension finie, chaque élément possède seulement un nombre fini de conjugués (autrement dit, un tel groupe est central au sens du rapporteur), la réciproque étant sans doute inexacte. Il serait intéressant de savoir si ce qui précède s'étend aux groups non discrets. *R. Godement* (Nancy).

Mautner, F. I. Infinite-dimensional irreducible representations of certain groups. Proc. Amer. Math. Soc. **1**, 582–584 (1950).

L'auteur donne un exemple extrêmement simple d'un groupe discret G pour lequel: (1) Tout élément de G n'a qu'un nombre fini de conjugués distincts; (2) il existe des représentations unitaires irréductibles de dimension infinie. Le groupe G est construit comme suit: Soit g le groupe $ax+b$ d'un corps fini; G est alors le groupe multiplicatif des suites d'éléments de g dont "presque tous" les termes sont égaux à l'élément unité de g, et G contient un sous-groupe abélien invariant évident, et les représentations cherchées de G sont construites à l'aide de caractères de ce sous-groupe. [Note du rapporteur. Le résultat précédent constitue un contre-exemple à un "théorème" annoncé par le rapporteur [C. R. Acad. Sci. Paris **225**, 221–223 (1947); ces Rev. **9**, 134], car parmi les "groupes centraux" du rapporteur figurent précisément les groupes discrets dans lesquels tout élément n'a qu'un nombre fini de conjugués.] Il y a lieu de modifier comme suit les résultats annoncés dans la note en question: Soit χ un caractère d'un groupe central G; alors il existe une double représentation unitaire irréductible $\{H, U_x, V_x, S\}$ de G telle que les U_x engendrent un facteur $\mathfrak{R}_l$ de classe finie, avec en outre $\chi(x)=\mathrm{tr}(U_x)$ où tr est la trace relative dans ce facteur; la formule de Plancherel annoncée par le rapporteur est encore exacte si l'on tient compte du fait que les traces qui y figurent sont prises dans des facteurs de classe finie, et non dans des représentations irréductibles. Précisons qu'un exposé correct de la théorie des groupes centraux avait été donné par H. Cartan à Chicago en janvier 1948, au cours d'un colloque sur la théorie des groupes; précisons aussi que le rapporteur n'a jamais publié un exposé détaillé de cette théorie parce qu'il avait l'espoir (d'ailleurs justifié) de la généraliser considérablement. *R. Godement.*

Mackey, George W. Functions on locally compact groups. Bull. Amer. Math. Soc. **56**, 385–412 (1950).

Cet article expose (évidemment sans démonstrations en général) les résultats les plus importants obtenus jusqu'en 1949 en ce qui concerne l'extension aux groupes localement compacts généraux de la transformation de Fourier. L'auteur traite tout d'abord des groupes compacts (formules de développement de Peter et Weyl), puis des groupes abéliens: groupe dual, formule de Plancherel, théorie des idéaux dans l'espace L^1 et théorèmes taubériens, transformation de Laplace. Ceci fait, l'auteur passe aux groupes généraux; après avoir rappelé le théorème de Gelfand et Raikov (existence d'un système complet de représentations unitaires irréductibles), l'auteur définit rapidement ce qu'est une "somme directe continue" d'espaces de Hilbert puis énonce le résultat récent de Mautner [voir l'analyse ci-dessus]: Toute représentation unitaire d'un groupe séparable se décompose en somme directe continue de représentations irréductibles; cette décomposition peut du reste ne pas être unique (même les composantes ne sont pas uniques), et l'auteur propose, pour rétablir une certaine unicité, de la remplacer par une décomposition en "factor representations" en utilisant une méthode de von Neumann (décomposition d'un anneau relativement à son centre). Cette méthode est particulièrement bien adaptée à l'étude des

représentations "régulières" du groupe G dans l'espace L^2 correspondant; dans ce cas, les anneaux $\mathfrak{R}_l$ et $\mathfrak{R}_r$ engendrés respectivement par les translations à gauche et à droite vérifient $\mathfrak{R}_l=(\mathfrak{R}_r)'$ de sorte que le centre de $\mathfrak{R}_l$ est engendré par les sous-espaces invariants à la fois à gauche et à droite; la méthode de décomposition précédente conduit donc à une décomposition simultanée des translations à gauche et à droite, et ceci conduit l'auteur, par analogie avec le cas des groupes compacts, à conjecturer l'existence d'une formule de Plancherel permettant de calculer le produit scalaire de deux éléments de L^2 à l'aide de traces relatives dans les facteurs de la décomposition, ce qui a été confirmé depuis (comme l'auteur le suggère, l'outil essentiel pour parvenir à ce résultat est la théorie des "H-systems" d'Ambrose, ou des "anneaux unitaires" de Rohlin, ou des "doubles représentations unitaires" du rapporteur; on peut de cette façon éviter complètement les "choix mesurables" qui interviennent dans la théorie générale de von Neumann, parce que dans tout anneau unitaire on peut définir explicitement et canoniquement une trace relative). Enfin, les difficultés qui s'opposent à une étude des idéaux dans L^1 sont signalées. L'article se termine par une bibliographie très complète et fort utile. *R. Godement* (Nancy).

Gleason, A. M. The structure of locally compact groups. Duke Math. J. **18**, 85–104 (1951).

This is a detailed exposition of results announced earlier [Proc. Nat. Acad. Sci. U. S. A. **35**, 384–386 (1949); these Rev. **10**, 678]. *K. Iwasawa* (Princeton, N. J.).

Mostow, G. D. A theorem on locally Euclidean groups. Proc. Amer. Math. Soc. **2**, 285–289 (1951).

Let G be a connected locally compact group and let G' denote the closure of the commutator group of G. Let the derived series of G be $G_0, G_1, \cdots, G_n, \cdots$, putting $G_0=G$ and $G_{n+1}=G'_n$. It is known that each G_n is connected and, for some n, $G_n=G_{n+1}=G_{n+2}=\cdots$. The author calls $C=G_n$ the core of G. The purpose of the present paper is to prove the following theorem: Let G be a connected locallyEuclidean group and let C be its core. Then G/C is a Lie group. For the proof of the theorem the author first proves the following: (1) Let A be a simply connected space and let f be a continuous mapping of A onto a locally connected space B. If $f^{-1}(b)$ is connected for all b in B, then B is simply connected. (2) A simply connected locally compact solvable group is a Lie group. The theorem is then proved by reducing to the case where G is simply connected. In that case, G/C is also simply connected by (1), and hence is a Lie group by (2). *K. Iwasawa* (Princeton, N. J.).

Gotô, Morikuni, and Yamabe, Hidehiko. On continuous isomorphisms of topological groups. Nagoya Math. J. **1**, 109–111 (1950).

A locally compact group G is called complete if the identity component of the group of continuous automorphisms of G (topologized in a certain natural manner) coincides with the group of inner automorphisms. Theorem: Let G be a locally connected complete group with compact center and let H be a locally compact group. A continuous isomorphism $\phi: G\rightarrow H$ such that $\phi(G)$ is everywhere dense in H is necessarily open and $\phi(G)$ is closed in H. The authors make a number of applications in the theory of (L)-groups [Iwasawa, Ann. of Math. (2) **50**, 507–558 (1949); these Rev. **10**, 679]. *P. A. Smith* (New York, N. Y.).

Malcev, A. I. On solvable topological groups. Amer. Math. Soc. Translation no. **42**, 14 pp. (1951).

Translated from Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 165–174 (1946); these Rev. **8**, 439.

Malcev, A. I. On a class of homogeneous spaces. Amer. Math. Soc. Translation no. **39**, 33 pp. (1951).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 9–32 (1949); these Rev. **10**, 507.

Matsushima, Yozô. On the decomposition of an (L)-group. J. Math. Soc. Japan **1**, 264–274 (1950).

L'auteur généralise aux L-groupes le théorème de décomposition de Levi et Malcev (rappelons qu'un L-groupe, par définition, admet un système complet de représentations dans des groupes de Lie). Le principal résultat obtenu est le suivant. Soit G un L-groupe connexe; on peut alors écrire $G=L\cdot C\cdot R$ où: (1) R est le radical de G (sous-groupe invariant fermé connexe résoluble maximal de G); (2) C est un sous-groupe compact connexe semi-simple de G; (3) L est un sous-groupe de Lie semi-simple de G, sans sous-groupe invariant compact connexe non trivial; de plus L commute avec C, et $L\cap C$ est un groupe fini; enfin, L et C sont déterminés à un automorphisme intérieur près défini par un élément de R. *R. Godement* (Nancy).

Dynkin, E. B. Maximal subgroups of semi-simple Lie groups and the classification of primitive groups of transformations. Doklady Akad. Nauk SSSR (N.S.) **75**, 333–336 (1950). (Russian)

The author announces the determination of all maximal (local) subgroups of (complex) semisimple Lie groups. This yields naturally a classification of the semisimple primitive analytic groups of transformations, and as the general case has been reduced to the simple case by Morosoff [Rec. Math. [Mat. Sbornik] N.S. **5(47)**, 355–390 (1939); these Rev. **1**, 258], this classification, first undertaken by Lie, can be regarded as complete. The present paper includes brief enumerations of the maximal subgroups of all the simple Lie groups. It is indicated that the proofs for the present results depend mainly on the determination of all subgroups of rank one in simple Lie groups, due to a variety of authors, and on recently published results of the present author [same Doklady (N.S.) **71**, 221–224 (1950); these Rev. **11**, 492]. *I. E. Segal* (Chicago, Ill.).

NUMBER THEORY

Thébault, Victor. À propos de carrés curieux. Mathesis **60**, 5–8 (1951).

Chandler, Albert. Benjamin Franklin's "Magical square of 16." J. Franklin Inst. **251**, 415–422 (1951).

Brewer, B. W. On the quadratic reciprocity law. Amer. Math. Monthly **58**, 177–179 (1951).

Moessner, Alfred. Some Diophantine problems with their solutions. Simon Stevin **27**, 196–200 (1950).

Gloden, A. Normal trigrade and cyclic quadrilateral with integral sides and diagonals. Amer. Math. Monthly **58**, 244–247 (1951).

Gloden, A. Sur une méthode inédite pour transformer en un carré une forme binaire du 4e degré. (Méthode des équations adjointes). Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. **19**, 239–242 (1950).

Given one rational solution of the equation $f(x)=y^2$, where $f(x)$ is a polynomial of degree 4 with rational coefficients, further solutions are found by means of an associated quadratic equation whose coefficients are quadratic in x and whose discriminant is $f(x)$. However, the problems of the existence and determination of the associated equation are not discussed. *I. Niven* (Eugene, Ore.).

Buquet, A. L'équation diophantienne

$$f(t)\equiv At^4+Bt^3+Ct^2+Dt+E=s^2$$

en nombres rationnels et les polygones de Poncelet. Mathesis **59**, 233–236 (1950).

From one rational solution of the given equation, Euler obtained an infinite sequence of rational solutions, in general all different. Euler's method employs a parameter, say m, and the sequence of solutions corresponds to a sequence of points (t, m) which form the (Poncelet) polygon mentioned in the title. Certain geometric properties of this polygon are studied, the details of which are omitted here. Reference should also be made to work of Mordell [Quart. J. Pure Appl. Math. **45**, 170–186 (1914)]. *I. Niven.*

Obláth, Richard. Über die diophantische Gleichung $x^3-1=2y^2$. Acta Math. Acad. Sci. Hungar. **1**, 113–117 (1950). (German. Russian summary)

It is proved that $x^3-1=3y^2$ and $x^3-1=2y^n$ ($n\geqq 2$) have no integral solutions with $|x|\neq 1$. Results of Nagell [Norsk Mat. Forenings Skr. (I) no. **2**, (1921)] and Ljunggren [Acta Math. **75**, 1–21 (1943); these Rev. **8**, 135] concerning the equations $x^2+x+1=3v^n$ and $x^2+x+1=v^n$ are employed. *I. Niven* (Eugene, Ore.).

Obláth, Richard. Über das Produkt fünf aufeinander folgender Zahlen in einer arithmetischen Reihe. Publ. Math. Debrecen **1**, 222–226 (1950).

Let $(a, d)=1$. The author proves that

$$a\cdot(a+d)(a+2d)(a+3d)(a+4d)=x^2$$

is impossible. *P. Erdös* (Aberdeen).

Früchtl, Kurt. Statistische Untersuchung über die Verteilung von Primzahl-Zwillingen. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1950**, 226–232 (1950).

The number of prime pairs $(p, p+2)$ between $1000n$ and $1000(n+1)$ is given for $n=0(1)1019$. It is pointed out that the number of primes is approximately ten times the number of prime pairs in a given interval. The number of prime quadruplets $(p, p+2, p+6, p+8)$ between 10^5n and $10^5(n+1)$ is given for $n=0(1)9$. The last members, $p+8$, not exceeding 1020000 are listed separately, 172 in all. Data are based on the old table of Chernac [Cribrum Arthimeticum, Daventriae, 1811] which has never been completely checked. *D. H. Lehmer* (Berkeley, Calif.).

Sierpinski, Waclaw. Un théorème sur les nombres premiers. Matematiche, Catania **5**, 66–67 (1950).

The following theorem is proved. For every integer n there exists a prime p such that the number of divisors of each integer of the sequence $p\pm k$, $k=1, \cdots, n$ is greater than n. From this follows a previous result of the author, namely that for every integer n there exists a prime $p>n$ such that each of the integers $p\pm k$ with $k=1, \cdots, n$ is composite [Colloquium Math. **1**, 193–194 (1948); these Rev. **10**, 431]. *W. H. Simons* (Vancouver, B. C.).

de Bruijn, N. G. On bases for the set of integers. Publ. Math. Debrecen **1**, 232–242 (1950).

A set of integers $\{b_1, b_2, \cdots\}$ is called a base for the set of all integers whenever any integer x can be expressed uniquely in the form $x=\sum_{i=1}^\infty \epsilon_i b_i$ ($\epsilon_i=0$ or 1, $\sum_{i=1}^\infty \epsilon_i<\infty$). It is proved that any base can, by rearrangement, be written in the form $\{d_1, 2d_2, 2^2d_3, \cdots\}$, where the d_i are odd, thus proving and extending a conjecture of Szele. The sequence $\{d_1, d_2, \cdots\}$ is then said to be basic, and remains basic whenever a finite number of odd numbers is added, omitted, or changed into other odd numbers. Various other properties of basic sequences are proved. A basic sequence is said to be periodic, with period s, if $d_{i+s}=d_i$ for all i. Necessary and sufficient conditions for periodic sequences to be basic are given, and basic sequences of period 2 are considered in greater detail. Such a sequence may be denoted by $[a, b]$ where $d_1=a$, $d_2=b$, and a and b must be of opposite signs. It is shown that if either a or b is divisible by any number of the form 2^m+1 then $[a, b]$ is not basic, and that the pair $[2^{2k+1}-1, -1]$ is basic for $k=1, 2, \cdots$. For $100\geqq a>-b>0$ there are 20 basic pairs $[a, b]$ which are listed. The problem is generalised by introducing the notion of an A-base. Let A be a finite set of h natural numbers $a_1, a_2, \cdots, a_h$ where $0\epsilon A$. The set $\{b_1, b_2, \cdots\}$ is called an A-base whenever each integer x can be written in the form $x=\sum_{i=1}^\infty \epsilon_i b_i$ for $\epsilon_i \epsilon A$, $\sum|\epsilon_i|<\infty$, and is called simple if it can be rearranged in the form $\{d_1, hd_2, h^2d_3, \cdots\}$. Some of the preceding results carry over to simple A-bases. The paper concludes with the discussion of three conjectures related to the problems considered. *R. A. Rankin* (Cambridge, England).

Apostol, T. M. Asymptotic series related to the partition function. Ann. of Math. (2) **53**, 327–331 (1951).

The author obtains an asymptotic expansion for the coefficients $a_p(n)$ generated by $f_p(x)=1+\sum_1^\infty a_p(n)x^n=\exp G_p(x)$, where $G_p(x)=\sum_1^\infty \sigma_p(n)n^{-p}x^n$ and $\sigma_p(n)=\sum_{d|n}d^p$, $p>1$. For $p=1$, the $a_p(n)$ reduce to $p(n)$, the number of unrestricted partitions of n. The asymptotic expansion consists of N terms of the usual structure with an error term $O(1)$ as $N\rightarrow\infty$. The $a_p(n)$ are rational numbers and represent certain finite sums of divisor functions. The author uses the classical Farey circle method in the form given by Rademacher in his treatment of $p(n)$ [Ann. of Math. (2) **44**, 416–422 (1943); these Rev. **5**, 35]. The behavior of $f_p(x)$ near each "rational" point of $|x|=1$ is obtained from the author's previous paper, which developed a transformation formula for that function [Duke Math. J. **17**, 147–157 (1950); these Rev. **11**, 641]. *J. Lehner* (Philadelphia, Pa.).

Rényi, Alfred. On a theorem of the theory of probability and its application in number theory. Časopis Pěst. Mat. Fys. **74** (1949), 167–175 (1950). (Russian. Czech summary)

The theorem of the theory of probability is the same as one given previously [J. Math. Pures Appl. (9) **28**, 137–149

(1949); Compositio Math. **8**, 68–75 (1950); these Rev. **11**, 161, 581]. It is applied here to prove the following result: Let $\Lambda(n)$ be the Mangoldt function and set $\psi(N)=\sum_{n=1}^{N}\Lambda(n)$, $\psi(N,p,r)=\sum_{n\leqq N,\,n\equiv r\,(\mathrm{mod}\,p)}\Lambda(n)$. Let $\frac{1}{3}\leqq\alpha<\frac{1}{2}$ and β,γ,δ positive numbers subject to the condition $\beta+\gamma+\delta<\alpha^{-1}-2$. Then, for all primes $p<N^{\alpha}$, except at most $N^{\alpha(1-\delta)}$, and all $r=1,2,\cdots,p-1$, except at most $p^{1-\gamma}$, one has $\psi(N,p,r)=\psi(N)/p+\theta\psi(N)/p^{1+\beta}$, where $|\theta|\leqq 1$.

M. Kac (Ithaca, N. Y.).

Shapiro, Harold N. On the iterates of a certain class of arithmetic functions. Comm. Pure Appl. Math. **3**, 259–272 (1950).

The author considers the class K of arithmetical functions $f(x)$ which satisfy the following three axioms. (A) For any two integers a, b, $f(ab)=f(a)f(b)g_f(d)$, where $d=(a,b)$, and $g_f(d)$ is a single-valued function of d, taking rational values. (B) There exists an integer $r>1$ such that for $x\leqq r$, $f(x)=1$, and for $x>r$, $f(x)\geqq r$. (C) For $x>r$, $f(x)<x$. From (B) and (C) it follows that, for all x, the iterates $f^k(x)$ form a decreasing sequence which for $x>r$ always reaches the value r. It is shown that $r=2$. Then, for each $n>1$, there is a unique k such that $f^k(n)=2$, and the function $C_f(n)$ is defined by $C_f(1)=0$, $C_f(n)=k$ $(n>1)$. When $f(x)$ is Euler's function $\varphi(x)$ the author [Amer. Math. Monthly **50**, 18–30 (1943); these Rev. **4**, 188] has already shown that $C_f(n)$ satisfies the property $C_f(ab)=C_f(a)+C_f(b)+\epsilon(d)$, where $\epsilon(d)$ is 0 or 1 according as $d=(a,b)$ is odd or even. For general $f\epsilon K$ he proves that a necessary and sufficient condition for $C_f(n)$ to satisfy this is that, for all primes p, $f(p)$ is even and $C_f(2f(p^2)/f(p))=C_f(p)$. This subclass of K is denoted by $\bar{K}$. Functions $f(x)$ satisfying (A) are shown to have a certain normal form, and the properties of transformations T of functions $f\epsilon\bar{K}$ such that $Tf\epsilon\bar{K}$ and $C_f(n)=C_{Tf}(n)$ are considered. Several other properties are proved and generalisations made which there is not space here to note.

R. A. Rankin (Cambridge, England).

Rédei, L. Über die Wertverteilung des Jacobischen Symbols. Acta Univ. Szeged. Sect. Sci. Math. **13**, 242–246 (1950).

Let m be an odd, square free integer, (x/m) the Jacobi symbol, and call $\sum_{\alpha\leqq x\leqq\beta}(x/m)$ the quadratic excess in the interval (α,β). Denote by $A_1\cdots A_8$, $B_1\cdots B_{10}$, $C_1\cdots C_{12}$ the quadratic excesses in the subintervals got by dividing $(0,m)$ into 8, 10, and 12 equal parts, respectively. Let h_k denote the class number of $R(\sqrt{-km})$. Dirichlet [J. Reine Angew. Math. **21**, 134–155 (1840)] and Gauss [Werke, v. 2, Göttingen, 1876, remarks by Dedekind pp. 301–303] found expressions for the A_i in terms of h_1 and h_2, and, when $3\nmid m$, for the C_i in terms of h_1, h_3. The author here derives an expression for the B_i in terms of h_1, h_5, in case $5\nmid m$ and $m\equiv 3 \bmod 4$.

G. Whaples (Bloomington, Ind.).

Grosswald, Emil. On the structure of some subgroups of the modular group. Amer. J. Math. **72**, 809–834 (1950).

In the group of modular substitutions under consideration the elements are square matrices of 4 elements of determinant unity. The subgroups are defined by imposing suitable supplementary conditions on the above 4 elements. These supplementary conditions are: 1 element zero, 2 elements zero, 2 elements zero and at the same time the other elements unity. Thus 4 subgroups are arrived at which it is the purpose of this paper to study. The main results of the paper are: The subgroups mentioned in the second place are generated by a number of independent generators of which the explicit form is found. A number of properties of these generators are given. If the modulus is 2, both independent generators are parabolic. If the modulus is 3 or more, 3 of the independent generators are parabolic. If the modulus is 2 or more, the genus of a fundamental region corresponding to the subgroups is zero. If the modulus is more than 3 the genus of the fundamental region is given by a simple equation. Finally, some relations between the principal subgroup and the above subgroups are stated.

M. J. O. Strutt (Zurich).

Eichler, Martin. Zur Algebra der orthogonalen Gruppen. Math. Z. **53**, 11–20 (1950).

The author first discusses briefly the similarities between the theory of simple algebras, the theory of algebraic number fields, and the theory of quadratic forms which he developed recently [Comment. Math. Helv. **20**, 9–60 (1947); **21**, 1–28 (1948); these Rev. **10**, 102], which have their basis in the rôles played in the theories by certain groups, for example, the orthogonal groups of the title in the theory of quadratic forms. He then proves theorems relating to quadratic forms which include a generalization and a specialization of theorems from the other theories. Let $\mathfrak{F}$ be the symmetric matrix, over a field k of characteristic not equal to 2, of a quadratic form $\frac{1}{2}\dot{x}\mathfrak{F}x$ [erroneously $\dot{x}\mathfrak{F}x$ in the review cited above] in n variables, which does not represent 0 properly in k [restriction erroneously omitted for certain conclusions in the second cited paper, §12]. A matrix $\mathfrak{J}$, over k or an extension field, is called $\mathfrak{F}$-orthogonal ($\mathfrak{F}$-o) if $\dot{\mathfrak{J}}\mathfrak{F}\mathfrak{J}=\mathfrak{F}$. The antiautomorphism $(\mathfrak{F})$: $\mathfrak{M}^{(\mathfrak{F})}=\mathfrak{F}^{-1}\dot{\mathfrak{M}}\mathfrak{F}$ of the ring (algebra over k) of all n by n matrices, is an automorphism of order 2 for certain commutative subrings R, called $\mathfrak{F}$-automorphic ($\mathfrak{F}$-a) rings, viz., such that both $\mathfrak{M}$ and $\mathfrak{M}^{(\mathfrak{F})}$ are in R. For example, $R=k\cdot\mathfrak{E}+k\cdot\mathfrak{J}+k\cdot\mathfrak{J}^2+\cdots$ is an $\mathfrak{F}$-a ring if $\mathfrak{J}$ is $\mathfrak{F}$-o. It is easily shown that any $\mathfrak{F}$-a ring is semisimple and can be imbedded in an $\mathfrak{F}$-a ring of rank n over k. Next, every $\mathfrak{F}$-o matrix $\mathfrak{J}$ of determinant 1 can be expressed in the form $\mathfrak{J}=(\mathfrak{M}^{(\mathfrak{F})})^{-1}\mathfrak{M}=\mathfrak{M}^{1-(\mathfrak{F})}$, where $\mathfrak{M}$ is a nonsingular element of an $\mathfrak{F}$-a ring. This follows from the fact that $\mathfrak{M}=\mathfrak{E}-(\mathfrak{E}+\mathfrak{J})^{-1}(\mathfrak{E}-\mathfrak{J})$ serves in case -1 is not an eigenvalue of $\mathfrak{J}$, together with a reduction of the other case to this by methods of the theory of matrices. A generalization of a theorem of Wedderburn is: If R_1 and R_2 are $(\mathfrak{F})$-operator-isomorphic $\mathfrak{F}$-o rings over k, there exists an $\mathfrak{F}$-o matrix $\mathfrak{S}$, with elements in k or in an extension of k, such that $R_2=\mathfrak{S}^{-1}R_1\mathfrak{S}$ (understood elementwise). The proof is quite elementary. A specialization of the norm-theorem for quadratic extensions of number fields is: If R_1 and R_2 in the foregoing theorem are of rank n over k, and if k is an algebraic number field, then there exists a matrix $\mathfrak{S}$ as described, with elements in k, if and only if the same is true for every $\mathfrak{p}$-adic extension of k. The proof of this is less elementary. The author suggests that the conclusions of the last two theorems may be valid under broader conditions than those which the restriction on $\mathfrak{F}$ imposes.

R. Hull.

Mills, W. H. The m-th power residue symbol. Amer. J. Math. **73**, 59–64 (1951).

Soit $(\alpha/A)_{k,m}$ le symbole des restes des puissances m-ièmes dans k ($\alpha\epsilon k$, A est un idéal de k tel que $(\alpha/A)_{k,m}$ soit défini), soit k un corps de nombres algébriques, contenant les racines m-ièmes de l'unité, et ne contenant, pour tout multiple propre m' de m divisant quelque puissance de m, aucune racine m'-ième primitive de l'unité. Alors, si $r\equiv 0 \pmod{m}$, et si K est une extension de k, contenant les racines r-ièmes

primitives de l'unité, l'auteur donne l'expréssion explicite de $(\alpha/A)_{K_r}$ comme puissance de $(\alpha/A)_{k_m}$. Ce résultat, dont la démonstration ne demande que des calculs triviaux, joue un role auxiliaire considérable dans un travail plus important de l'auteur [voir l'analyse ci-dessous]. *M. Krasner.*

Mills, W. H. Reciprocity in algebraic number fields. Amer. J. Math. **73**, 65–77 (1951).

Dans ce travail, l'auteur résout complètement le problème de l'expréssion explicite du symbole des restes normiques (α, β, k, m) de Hilbert (où k èst un corps de nombres $\mathfrak{p}$-adiques contenant les racines m-ièmes de l'unité, où $\alpha, \beta \epsilon k$, et où m est un entier rationnel). En vertu des propriétés bien connus de ce symbole, il suffit de donner cette exprés-sion quand m est une puissance p^n de la caractéristique p du corps résiduel de k, $\beta=\pi$ définit l'idéal premier $\mathfrak{p}$ de k, $\alpha\equiv 1 \pmod{\mathfrak{p}}$, et k ne contient aucune racine p^{n+1}-ième primitive de l'unité.

Soit k_0 le corps p-adique rationnel, k^* l'extension de k, obtenue en lui adjoignant toutes les puissances fraction-naires de $\beta=\pi$, $S(\xi)$ la fonction, linéaire par rapport à k_0, définie sur k^*, qui coïncide avec la trace $S_{k/k_0}(\xi)$ si $\xi \epsilon k$, et est égale à 0 si $\xi=\gamma\pi^r$, où $\gamma\epsilon k$, et où r n'est pas entier. Supposons que, si l'on adjoint à k les racines p^{n+1}-ièmes de l'unité, le corps ainsi obténu contient les racines primitives $p^{n+1+\delta}$-ièmes de l'unité, sans contenir de telles racines $p^{n+2+\delta}$-ièmes de l'unité (d'ailleurs, $\delta=0$ si $p^n>2$). Soit $\eta=+1$ ou -1, suivant que $p^n>2$ ou $=2$. Soient ζ une racine p^n-ième primitive de l'unité, et ρ une racine $p^{n+1+\delta}$-ième primitive de l'unité. Soit $s=cp^v$, où $p \nmid c$. Posons

$$F_\zeta(s)=\begin{cases} -\frac{1}{2}-1/(\zeta^c-1), & \text{si} \quad p^n>2,\ v\geqq n,\\ 1/(\zeta^s-1)-1/(\zeta^c-1), & \text{si} \quad p^n>2,\ v<n,\\ \frac{1}{2}(-1)^{2^{v-1-\delta}c}, & \text{si} \quad p^n=2,\ v>\delta,\\ 0, & \text{si} \quad p^n=2,\ v=\delta,\\ S_{k(\rho)/k}(\rho^s/(\rho^{4s}-1)), & \text{si} \quad p^n=2,\ v<\delta; \end{cases}$$

$\mu(s)$ étant la fonction de Möbius, posons

$$\mu^*(s)=\begin{cases} \mu(s), & \text{si} \quad p^n>2,\\ -\mu(s), & \text{si} \quad p^n=2 \text{ et } v=0,\\ -2^{v-1}\mu(s), & \text{si} \quad p^n=2 \text{ et } v>0. \end{cases}$$

Posons encore

$$\mathfrak{F}_\zeta(s)=E\textstyle\sum_{d|s}F_\zeta(s/d)\mu^*(d)d^{-1}, \quad \text{où} \quad E=1 \quad \text{ou} \quad 1+2^{n-1},$$

suivant que $\rho\neq$ ou $=2$. Soient f le degré résiduel absolu de k et w une racine (p^f-1)-ième primitive de l'unité.

Considérons le corps $\bar{Q}=k_0(x, y)$ des polynomes en x, y dans k_0, organisé en corps valué par la valuation, pro-longeant celle de k_0, telle que $|x|=1$, $|y|=|\pi|=|\mathfrak{p}|$, et $|\sum a_{i,j}x^iy^j|=\max_{i,j}|a_{i,j}x^iy^j|$. Soient Q le complété de $\bar{Q}$ par rapport à cette valuation, et Q^* l'extension valuée de Q, obtenue en lui adjoignant toutes les puissances fraction-naires de y. Puis $f(x, y)\rightarrow f(\eta w, \pi)$ est un homomorphisme uniformément continu de $\bar{Q}$ dans k (car il n'augmente pas la valuation), et l'image de $\bar{Q}$ par cet homomorphisme est dense sur k. Par suite, $f(x, y)\rightarrow f(\eta w, \pi)$ est défini pour tout $f(x, y)\epsilon Q^*$, et est un homomorphisme de Q^* sur k^*, qui applique Q sur k.

Si $H(x, y)\epsilon Q$ est une série de Taylor en x, y à coefficients entiers rationnels, qui soit un polynome par rapport à x, et si $G(x, y)=1+yH(x, y)$, on a $|G(x, y)-1|\leqq|\pi|<1$. Par suite, pour tout nombre rationnel r, $\log H(x, y^r)=\sum_{s=1}^{\infty}H(x, y^r)^s/s$ est défini, et est de la forme $\sum_1^\infty b_i(x)y^{ri}$ avec $|b_i(x)|\leqq|i|^{-1}$. Par suite, la série $y\int_0^y \log G(x, y^r)dy=\sum_1^\infty b_i(x)(ri)^{-1}y^{ri}$ con-verge dans Q^*, et sa valeur pour $x=\eta w$ et $y=\pi$ est un élément de k^*. D'autre part, quelque soit $\alpha\equiv 1 \pmod{\mathfrak{p}}$ de k, il existe une série $G(x, y)$ de la forme indiquée telle que $\alpha=G(\eta w, \pi)$. On appelera de telles $G(x, y)$ représentations normales de α.

Ceci posé, le résultat de l'auteur est que, si $G(x, y)$ est une réprésentation normale de $\alpha\equiv 1 \pmod{\mathfrak{p}}$, on a $(\alpha, \pi, k, p^n)=\zeta^B$, où $B=$

$$S\left(\tfrac{1}{2}(\eta+1)\log\alpha+\sum_1^\infty\mathfrak{F}_\zeta(s)\left[y\int_0^y \log G(x, y^{sp^{-n-\delta}})\right]_{x=\eta w,\, y=\pi}\right).$$

La démonstration se fait par des calculs trop longs pour être exposés ici en détail, mais dont l'idée centrale est ingénieuse: On décompose α en un produit de facteurs de la forme $1-\omega\pi^{ap^r}$, où $p\nmid a$, et où ω est une racine (p^f-1)-ième de l'unité; α étant de cette forme, on remplace, grâce au résultat du travail précédent de l'auteur [voir l'analyse ci-dessus], (α, π, k, p^n) par une puissance de $(\alpha, \pi, K, p^{n'})$, où $n'=\max(n+\delta, r)$, et où K s'obtient en adjoignant à k ou à $k(i)$, suivant que $p^n>$ ou $=2$, une racine p^r-ième primitive de l'unité ϵ; α se décompose dans K en produit de facteurs $\alpha_\nu=1-\omega_0\epsilon^\nu\pi^a$, où ω_0 est la racine (p^f-1)-ième de l'unité, telle que $\omega_0^{p^r}=\omega$; puisque

$$\begin{aligned}1=(\alpha_\nu, 1-\alpha_\nu, K, p^{n'})\\ =(\alpha_\nu, \omega_0, K, p^{n'})(\alpha_\nu, \epsilon, K, p^{n'})^\nu(\alpha_\nu, \pi, K, p^{n'})^a,\end{aligned}$$

et puisque $(\alpha_\nu, \omega_0, K, p^{n'})=1$, le calcul de $(\alpha_\nu, \pi, K, p^{n'})$ se réduit (car $a\not\equiv 0 \pmod{p}$) à celui de $(\alpha_\nu, \epsilon, K, p^{n'})$; en vertu d'un résultat bien connu d'Artin et Hasse, $(\alpha_\nu, \epsilon, K, p^{n'})=\epsilon^A$, avec $A=E'_p{}^{-n'}S_{k/k_0}(\log\alpha)$ (où $E'=1$ où $1+2^{n'-1}$, suivant que $p\neq$ ou $=2$). Les autres calculs sont purement téchniques. *M. Krasner* (Paris).

Kanold, Hans-Joachim. Sätze über Kreisteilungspoly-nome und ihre Anwendungen auf einige zahlentheo-retische Probleme. I. J. Reine Angew. Math. **187**, 169–182 (1950).

In an earlier paper [same J. **183**, 98–109 (1941); these Rev. **3**, 268] the author studied the prime divisors of the cyclotomic polynomials $F_m(x)$. In this paper similar results are obtained for the prime divisors of the binary forms $G(x, y)=y^{\varphi(m)}F_m(x/y)$. If $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}>2$ with $p_1<p_2<\cdots<p_k$, then all the prime divisors of $G(x, y)$ are congruent to 1 (mod m) if $p_k\not\equiv 1$ (mod $m/p_k^{\alpha_k}$). For $p_k\equiv 1$ (mod m/p_k) p_k can also be a divisor, but not p_k^2. Moreover, let n be a product Π of primes of form $mz+1$, or $p_k\Pi$ if $p_k\equiv 1$ (mod $m/p_k^{\alpha_k}$). The author determines the number of integers x for which $0<x<n$ and for which n is a divisor of $F_m(x)$. As application it is proved that for every prime $q>23$ and for every r the least rth power nonresidue mod q is less than $\sqrt{q}$. This result is known and can be obtained more simply [see the reviewer, Math. Z. **33**, 161–176 (1931); **35**, 39–50 (1932), in particular, p. 42]. *A. Brauer.*

Wang, Shianghaw. An existence theorem for abelian extension over algebraic number fields. Acad. Sinica Science Record **3**, 25–27 (1950). (English. Chinese summary)

The corrected version of Grunwald's theorem on the existence of cyclic extensions of an algebraic number field F with prescribed completions at a finite set S of primes, as formulated and proved in a previous paper by the author [Ann. of Math. (2) **51**, 471–484 (1950); these Rev. **11**, 489] is generalized to a theorem on the existence of Abelian extensions with prescribed Galois group G and with a finite set of prescribed completions whose Galois groups are imbeddable in G. The problem reduces immediately to the case where the desired extension is to have prime power degree l^r. It follows easily from the former theorem that

an extension with the desired properties exists unconditionally, except under the following circumstances: Let t denote the largest integer ν for which $\cos(2\pi/2^\nu)\in F$. Let s denote the largest positive integer x for which at least one of the given local extensions has a subextension of degree 2^x in which $[\cos(2\pi/2^{t+1})]^{2^x}$ is not a norm, if this occurs, and let $s=1$ otherwise. Then the exceptional case is the following: $l=2$, and $F(e^{2\pi i/2^s})$ is not cyclic over F, and S contains all oddly even primes, i.e., all primes $\mathfrak{p}$ dividing 2 and such that $[F_\mathfrak{p}(e^{2\pi i/2^x}):F_\mathfrak{p}]=[F(e^{2\pi i/2^x}):F]$, for all positive integers x.

Necessary and sufficient conditions for the existence of an extension K/F with Galois group G and prescribed completions $K^\mathfrak{p}F_\mathfrak{p}$ for $\mathfrak{p}\in S$ are determined for the exceptional case. The result is the following: Decompose G into a direct product of cyclic groups, arranged in a sequence with decreasing orders (which are then the uniquely determined elementary divisors). Decompose each $K^\mathfrak{p}$ similarly into cyclic components. Now let m denote the number of components of G whose orders are greater than 2^s, and let k denote the number of components of G whose orders are exactly 2^s. Let $m_\mathfrak{p}$ be the number of components of $K^\mathfrak{p}$ whose degrees are greater than 2^s. Finally, let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ be all the oddly even primes $\mathfrak{p}$ for which $K^\mathfrak{p}$ has a subfield of degree 2^s over $F_\mathfrak{p}$ in which $[\cos(2\pi/2^{t+1})]^{2^s}$ is not a norm. Then the desired extension K/F exists if and only if either $m_{\mathfrak{p}_i}<m$, for some $i=1, \cdots, h$, or h is even, or h and k are both greater than 1. *G. Hochschild* (Urbana, Ill.).

Kuroda, Sigekatu. Über die Klassenzahlen algebraischer Zahlkörper. Nagoya Math. J. 1, 1–10 (1950).

Let k be an algebraic number field and let K be a normal field over k. There exist a number of relations of the form $\prod\zeta(s,\Omega)^{a(\Omega)}=1$ with integral rational exponents between the zeta-functions of the fields Ω with $k\leqq\Omega\leqq K$; these relations have been discovered by E. Artin [Abh. Math. Sem. Hamburg Univ. **3**, 89–108 (1923); **8**, 292–306 (1930)]. Let $h(\Omega)$ denote the number of classes of ideals of Ω, let $w(\Omega)$ denote the number of roots of unity in the field Ω, and let $R(\Omega)$ denote the regulator of Ω. It follows that $\prod h(\Omega)^{a(\Omega)}\cdot\prod R(\Omega)^{a(\Omega)}=\prod w(\Omega)^{a(\Omega)}$. [The same formula forms the starting point of the paper reviewed below.] For relatively Abelian fields K/k, one of Artin's relations is given explicitly. In particular, if the Galois group is of prime power order l^m and Abelian of type $(l, l, \cdots, l)$, the equation holds $H/h=A\prod_j(h(\Omega_j)/h)$, where $H=h(K)$, $h=h(k)$, Ω_j ranges over the $(l^m-1)/(l-1)$ subfields of K of degree l over k, and where A has the value

$$(R(k)/R(K))(w(K)/w(k))\prod_j(R(\Omega_j)R(k)^{-1}w(k)w(\Omega_j)^{-1}).$$

It follows already from the results of H. Nehrkorn [Abh. Math. Sem. Hamburg. Univ. **9**, 318–334 (1933)] that A is a power of l; the author obtains an explicit formula for the value of A. This result can be considered as a generalization of the theorem of Dirichlet and Herglotz [cf. G. Herglotz, Math. Z. **12**, 255–261 (1922)]. *R. Brauer.*

Brauer, Richard. Beziehungen zwischen Klassenzahlen von Teilkörpern eines galoisschen Körpers. Math. Nachr. **4**, 158–174 (1951).

With a cyclic subgroup $\mathfrak{Z}$ of a finite group $\mathfrak{G}$, let $\psi_\mathfrak{Z}$ be the character of $\mathfrak{G}$ induced by the principal character of $\mathfrak{Z}$. Every rational character Φ of $\mathfrak{G}$ is a linear combination of $\psi_\mathfrak{Z}$ with some rational coefficients $c_\mathfrak{Z}$, $\mathfrak{Z}$ running over cyclic subgroups of $\mathfrak{G}$. The present paper starts by determining a certain explicit set of $c_\mathfrak{Z}$, as follows

$$(1)\qquad c_\mathfrak{Z}=(\mathfrak{G}:\mathfrak{Z})^{-1}\textstyle\sum_{z^*}\mu((\mathfrak{Z}^*:\mathfrak{Z}))\Phi(z^*),$$

where the sum is extended over all cyclic subgroups $\mathfrak{Z}^*$ of $\mathfrak{Z}$, with generators z^*, and μ is the Möbius function. Let K be an algebraic number field normal over the rational field P, and $\mathfrak{G}$ be its Galois group. In order that a relation $\prod_\Omega\zeta(s,\Omega)^{a(\Omega)}=1$ of the ζ-functions of subfields Ω holds, it is necessary and sufficient that $\sum a(\Omega)\psi_\mathfrak{H}=0$, where $\mathfrak{H}$ denotes the subgroup of $\mathfrak{G}$ belonging to Ω and $\psi_\mathfrak{H}$ is the character of $\mathfrak{G}$ induced by the principal character of $\mathfrak{H}$. The above result, applied to $\mathfrak{H}$ belonging to a (fixed) Ω, in place of $\mathfrak{G}$, gives

$$(2)\quad \zeta(s,\Omega)=\textstyle\prod_Z\zeta(s,Z)^{c(Z)}\quad\text{with}\quad c(Z)=[Z:\Omega]^{-1}\sum_{Z^*}\mu([Z:Z^*]),$$

where Z runs over intermediate fields of K/Ω over which K is cyclic and where Z^* runs over subfields of Z over which K is cyclic. In fact, every relation among the ζ-functions of the subfields of K is a consequence of the present one (with different Ω's). From Hecke's formula for the residue of a ζ-function at its pole $s=1$, we see that a product-relation (with exponents $a(\Omega)$ which we assume to be integral) among the ζ-functions implies the relation

$$(3)\qquad \textstyle\prod_\Omega(R(\Omega)h(\Omega))^{a(\Omega)}=\prod_\Omega w(\Omega)^{a(\Omega)},$$

where $R(\Omega)$, $h(\Omega)$ are respectively the regulator, and the class-number of Ω, and $w(\Omega)$ is the number of roots of unity in Ω. It is shown that the right-side is a power of 2. If our relation is in particular (2) with a fixed $\Omega=\Omega_0$, then it is 1 or $2^{1-(4l/v)}$ according as the highest power v of 2 dividing $w(K)$ is $=$ or >2, where l is the degree (over P) of the intersection of Ω_0 and the field of vth roots of unity.

The effect of an element α of $\mathfrak{G}$ on a fundamental system of units can be expressed by a matrix $\mathfrak{M}^*(\alpha)=\begin{pmatrix} m_q(\alpha) & 0 \\ q(\alpha) & \mathfrak{M}(\alpha)\end{pmatrix}$, where the first column $\binom{m_q}{q}$ is to be considered modulo $w=w(K)$. The representation $\mathfrak{M}(\alpha)$ of $\mathfrak{G}$ is equivalent to the representation $\mathfrak{L}$ induced by the unit representation of the subgroup $\mathfrak{C}$ belonging to the maximal real subfield K_0 of K, minus the unit representation. Let $\mathfrak{L}X=X\mathfrak{M}$ with a rational integral matrix X. It is shown that for a subfield Ω, belonging to $\mathfrak{H}$, $u(\mathfrak{H})R(\Omega)=R_\eta$, where R_η is the regulator of a certain system of units in Ω derived from a Minkowski unit θ of K, and $u(\mathfrak{H})$ depends only on the groups $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{C}$, and the (integral equivalence) class of $\mathfrak{M}^*$ and the matrix X. By means of the orthogonality relations of regular and induced representations, R_η is shown to have the form $[\Omega:P]^{-1}\prod\varphi_\kappa^{q_\kappa}$ if K is real, and similarly otherwise, where q_κ is the multiplicity of the unit character of $\mathfrak{H}$ in the contraction to $\mathfrak{H}$ of the κth irreducible character of $\mathfrak{G}$ and φ_κ is the product of characteristic roots of a certain matrix depending on θ, $\mathfrak{G}$, $\mathfrak{C}$ (and κ) only. Letting Ω vary over subfields of K, we put this result into (3). Observing the relation

$$\textstyle\sum_\Omega a(\Omega)q_\kappa(\Omega)=\sum a(\Omega)\psi_\mathfrak{H}(1)=0,$$

we obtain

$$(4)\qquad \textstyle\prod_\Omega h(\Omega)^{a(\Omega)}=\prod_\Omega w(\Omega)^{a(\Omega)}\prod_\Omega[\Omega:P]^{a(\Omega)}J,$$

where J is the product $\prod_\Omega u(\mathfrak{H})^{a(\Omega)}$ and turns out to depend only on $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{C}$, and the (integral equivalence) class of M^* (but not on X). (The first product in the right-side has been determined above.) The result gives a generalization and perfection of the results of Dirichlet, Hilbert, Herglotz, and Nehrkorn. A certain portion of the paper is in close connection with one by Kuroda [see the preceding review] (which is however devoted mainly to the (relative-) Abelian case). *T. Nakayama* (Urbana, Ill.).

Hasse, Helmut. Bemerkungen zu den Ring- und Strahlklasseneinteilungen in quadratischen Zahlkörpern. Math. Nachr. **4**, 322–327 (1951).

The author establishes some elementary facts on the conductors of "proper" ray class characters (i.e., the cosets of the corresponding class groups are not already composed of ring classes with respect to a rational module) in quadratic fields (for example, the conductor of such a "proper" ray class group does not divide $2p_\infty$) and the distribution of the units in the rays. These results are needed for the evaluation of the L-series belonging to characters of real quadratic fields. *O. F. G. Schilling* (Chicago, Ill.).

Davenport, H. Euclid's algorithm in cubic fields of negative discriminant. Acta Math. **84**, 159–179 (1950).

The author's basic result is as follows. Of the linear forms: $\xi=\alpha u+\beta v+\gamma w$, $\xi'=\alpha' u+\beta' v+\gamma' w$, $\xi''=\alpha'' u+\beta'' v+\gamma'' w$, suppose the coefficients α, β, γ are real, α', β', γ' are complex, and $\alpha'', \beta'', \gamma''$ are the conjugates of α', β', γ', respectively. Let the determinant of the forms be $i\Delta\neq 0$, and suppose, without loss of generality, $\Delta>0$. Denote the adjoint linear forms (with coefficients given by the inverse matrix) by: $\Xi=\mathrm{A}U+\mathrm{B}V+\Gamma W$, $\Xi'=\mathrm{A}'U+\mathrm{B}'V+\Gamma'W$, $\Xi''=\mathrm{A}''U+\cdots$. Their determinant is $(i\Delta)^{-1}$. Write $f(u, v, w)=\xi\xi'\xi''$. Then if no one of Ξ, Ξ', Ξ'' represents zero for integral values not all zero of U, V, W, there exist real numbers u^*, v^*, w^*, such that $|f(u+u^*, v+v^*, w+w^*)|\geqq c\Delta$, for all integers u, v, w, where c is a positive absolute constant. To prove this, the author first applies a classical theorem of Gauss on minima (lemma 1), and classical arguments of Hermite, to the positive definite ternary quadratic form

$$Q_R(U, V, W)=R^2X^2+2R^{-1}(Y^2+Z^2),$$

of determinant $D=4\Delta^{-2}$, where $X=\Xi$, $Y+iZ=\Xi'\sqrt{2}$, $Y-iZ=\Xi''\sqrt{2}$, and where R is a positive number. He obtains lemma 2: There exists a "chain" (i.e., n ranges over all integers: $-\infty<n<\infty$) of values x_n, y_n, z_n, of X, Y, Z, given by integral values of U, V, W, with the properties (a) for each n there is an $R=R_n>0$ such that $R_n^2x_n^2+R_n^{-1}(y_n^2+z_n^2)\leqq R_n^2X^2+R_n^{-1}(Y^2+Z^2)$ for all integers U, V, W not all zero; (b) for every n, $x_n>0$, $x_{n+1}<x_n$, $y^2_{n+1}+z^2_{n+1}>y_n^2+z_n^2$, $x_n(y^2_{n+1}+z^2_{n+1})\leqq\Delta^{-1}\sqrt{2}$; (c) $x_n\to 0$ and $y_n^2+z_n^2\to\infty$ as $n\to\infty$, $x_n\to-\infty$, $y_n^2+z_n^2\to 0$ as $n\to-\infty$. Secondly (lemmas 3, 4, 5), he proves that from such a chain one can select a chain such that, for any given $C>1$, (a) holds with k replacing n, k ranging over all integers, and X_k, Y_k, Z_k replacing x_n, y_n, z_n, while in (b) and (c) the stronger conditions hold that, for all k, $X_k>0$, $X_k\geqq CX_{k+1}$, $Y^2_{k+1}+Z^2_{k+1}\geqq C^2(Y_k^2+Z_k^2)$, $X_k(Y^2_{k+1}+Z^2_{k+1})<\sqrt{2}\Delta^{-1}C^6$. For the next lemmas let

$$Q_k=Q_{R_k}(U, V, W)=R_k^2X^2+R_k^{-1}(Y^2+Z^2)$$

be transformed, for each k, by an integral unimodular substitution on U, V, W into the form $\mathfrak{A}_kU_k^2+\mathfrak{B}_kV_k^2+\mathfrak{C}_kW_k^2+\cdots$, with new variables U_k, V_k, W_k, with

$$\mathfrak{A}_k=R_k^2X_k^2+R_k^{-1}(Y_k^2+Z_k^2),$$

that is, the minimum of Q_k by the foregoing, and with $\mathfrak{A}_k\leqq\mathfrak{B}_k\leqq\mathfrak{C}_k$, $\mathfrak{A}_k\mathfrak{B}_k\mathfrak{C}_k\leqq 2D=8\Delta^{-2}$. Under the kth substitution let Ξ, Ξ', Ξ'' become $\Xi=\mathrm{A}_kU_k+\mathrm{B}_kV_k+\Gamma_kW_k$, etc., with matrix whose inverse defines the corresponding small Greek letters $\alpha_k, \cdots$. Under the substitution contragredient to the kth one, ξ, ξ', ξ'' become $\xi_k=\alpha_ku_k+\beta_kv_k+\gamma_kw_k$, etc. The hypothesis of the basic result implies that none of $\mathrm{A}_k, \cdots, \Gamma_k''$ is zero. Lemma 6 states that, for all k, $\mathrm{A}_k>0$, $\mathrm{A}_k\geqq C\mathrm{A}_{k+1}$, $|\mathrm{A}'_{k+1}|\geqq C|\mathrm{A}_k'|$, $|\mathrm{A}_k\mathrm{A}'_{k+1}\mathrm{A}''_{k+1}|<\Delta^{-1}C^6/\sqrt{2}$, which is a restatement of lemma 5. Lemma 7 states that $|\mathrm{A}_k\beta_k|\leqq 1/\sqrt{2}$, $|\mathrm{A}_k\gamma_k|\leqq 1/\sqrt{2}$, $|\mathrm{A}_k'\beta_k'|\leqq 3/2\sqrt{2}$, $|\mathrm{A}_k'\gamma_k'|\leqq 3/2\sqrt{2}$. Lemma 8 states that, for every integer k, there exist integers p_k, q_k, such that $1/2\sqrt{2}<\mathrm{A}_k(p_k\beta_k+q_k\gamma_k)\leqq 1/\sqrt{2}$,

$$|\mathrm{A}_k'(p_k\beta_k'+q_k\gamma_k')|\leqq 9/2\sqrt{2}.$$

Finally the basic result is proved by showing that, if C is taken sufficiently large, the quantities $\lambda_0=\sum_{r=-\infty}^{0}(p_r\beta_r+q_r\gamma_r)$, $\lambda_0'=-\sum_{r=1}^{\infty}(p_r\beta_r'+q_r\gamma_r')$, $\lambda_0''=-\sum_{r=1}^{\infty}(p_r\beta_r''+q_r\gamma_r'')$ yield u^*, v^*, w^* via an integral unimodular substitution on u^*, v^*, w^* which are determined by the equations

$$\alpha_0u_0^*+\beta_0v_0^*+\gamma_0w_0^*=\lambda_0,\quad \alpha_0'u_0^*+\cdots=\lambda_0',$$
$$\alpha_0''u_0^*+\cdots=\lambda_0''.$$

An addition to the basic result is needed for its application to the algorithm question, viz., that if $f(u, v, w)$ has integral coefficients and does not vanish for integral u, v, w not all zero, then rational u^*, v^*, w^* exist. The proof of this involves lemmas 9–14. Then it follows easily that a cubic field with one real and two complex conjugates, for which $f(u, v, w)$ becomes the norm-form in the application, cannot be Euclidean if its discriminant d satisfies $d>c^{-2}$. A legitimate value of c^{-1} is $8(10)^{18}$, corresponding to an admissible choice of $C=37.5$ above. Hence, by a theorem of Minkowski, only a finite number of the cubic fields in question are Euclidean. The author has proved a similar basic result applicable to real quadratic fields [Quart. J. Math. Oxford Ser. (2) **1**, 54–62 (1950); these Rev. **11**, 582; cf. Hua's review of a paper by Inkeri, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **41** (1947); these Rev. **10**, 15]. See the following review for a quartic case. *R. Hull* (Lafayette, Ind.).

Davenport, H. Euclid's algorithm in certain quartic fields. Trans. Amer. Math. Soc. **68**, 508–532 (1950).

The title refers to quartic fields K with two pairs of complex conjugates. The basic result of the paper has to do with the product of four linear forms in four variables, which becomes the norm-form of a K in the application of an addition to the basic result. The essential ideas are the same as those of the paper on cubic fields reviewed above, with quaternary quadratic forms replacing ternary ones, etc., and with the addition of a new "first process of selection." The new process is employed between the corresponding stages to those of lemma 2 and lemmas 3–5 for the cubic case. Again the conclusion is that only a finite number of the fields in question are Euclidean, completing the cases of fields with one fundamental unit. The author says that so far he has been unable to extend his work to other fields. *R. Hull* (Lafayette, Ind.).

Poitou, Georges, et Descombes, Roger. Sur l'approximation dans le corps des racines cubiques de l'unité. C. R. Acad. Sci. Paris **232**, 292–294 (1951).

In a previous note [same C. R. **231**, 264–266 (1950); these Rev. **12**, 162] the authors have sketched briefly a theory of continued fractions applicable in each of the five Euclidean imaginary quadratic fields $R(im^{\frac{1}{2}})$, $m=1, 2, 3, 7$, and 11, and their successful application of the theory, in the case $m=11$, to determining the inf C of $C(x)$, as x ranges over all complex numbers, for the "constants of approximation": $C(x)=\lim\sup_{p,q}|q(p-qx)|^{-1}$, as p and q range over the integers of $R(im^{\frac{1}{2}})$. In the present note, they sketch some applications of the theory in the case $m=3$. This case is unique among the five in having two properties in common with the rational case which simplify the study of $C(x)$. They report that they are able to establish (1) Perron's

value $C=13^{1/4}=1.8988\ldots$, for the "first" constant of approximation, i.e., Hurwitz' constant, of $K=R(i3^{1/2})$; (2) that C is isolated among the values of $C(x)$ for K and corresponds to certain values of x which are quadratic over K; (3) that the second value of $C(x)$ for K is 2, which is also isolated and also corresponds to certain relative quadratic values of x; (4) the third value of $C(x)$ for K is $(32(3)^{1/2}/13)^{1/2}=2.06487\ldots$, which is also isolated, etc.; (5) every other value of $C(x)$ for K exceeds 2.070068; (6) $((28+16(3)^{1/2})/13)^{1/2}=2.0701693\ldots$, is a point of accumulation of values of $C(x)$ for K. *R. Hull.*

Linnik, Yu. V. An elementary method for a problem of the theory of prime numbers. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 198 (1950). (Russian)

The author suggests a possible method for proving elementarily Siegel's theorem on the class-number of positive definite binary quadratic forms [Acta Arith. **1**, 83–86 (1935)]. He has since constructed a proof on these lines [Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 327–342 (1950); these Rev. **12**, 482]. *H. Davenport* (London).

Obrechkoff, N. Sur l'approximation de n formes linéaires à n inconnues. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **45**, 287–292 (1949). (Bulgarian. French summary)

Let $f_1, \cdots, f_m$ be real linear forms in $x_1, \cdots, x_m$, and let n be a positive integer. It follows at once from the consideration of differences (mod 1) that there exist integers $x_1, \cdots, x_m$, not all zero, with $|x_i|\leqq n$, such that $|f_i-y_i|\leqq(n+1)^{-1}$ for $i=1, \cdots, m$, where $y_1, \cdots, y_m$ are also integers. The author proves that the sign of equality is needed only if $f_i=(b_{i1}x_1+\cdots+b_{im}x_m)/(n+1)$, where the b_{ij} are integers and their determinant is relatively prime to $n+1$. *H. Davenport* (London).

Obrechkoff, N. Sur l'approximation des nombres irrationnels. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **45**, 179–201 (1949). (Bulgarian. French summary)

The author's first result is a simple generalization of Borel's theorem on three successive convergents to a continued fraction. Let

$$\theta=a_0+\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots,$$

and let p_n/q_n be the general convergent to θ. Then the inequality $|\theta-p_i/q_i|<1/q_i^2(a_n^2+4)^{1/2}$ is satisfied for at least one of the three values $n-2$, $n-1$, n of i. The other results concern the cases of equality in various inequalities satisfied by linear forms. For example, if $\omega_1, \cdots, \omega_m$ are any real numbers, and n is any positive integer, it is obvious that there exist integers $x_1, \cdots, x_m$, not all zero, with $|x_i|\leqq n$, such that $|\omega_1x_1+\cdots+\omega_mx_m-y|\leqq(n+1)^{-m}$, where y is also an integer. The author gives the forms of $\omega_1, \cdots, \omega_m$ for which the sign of equality is necessary. *H. Davenport.*

Fel'dman, N. I. The approximation of certain transcendental numbers. I. Approximation of logarithms of algebraic numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 53–74 (1951). (Russian)

The author proves a number of important results on the approximation of the logarithm of an algebraic number by means of algebraic numbers of both variable degree and variable height; for instance: There exists a positive constant γ such that

$$|P(\pi)|>\exp\{-\gamma n(n\log n+1+\log H)\times\log(n\log n+2+\log H)\}$$

for every polynomial $P(z)\neq 0$ of degree n with rational integral coefficients of absolute value not greater than H. The proofs are given with full details. They depend on the study of polynomials in z and e^z that, together with their derivatives up to a high order, vanish at a great number of points of an arithmetic progression, and are based on the ideas used by Gel'fond in his classical proof of the transcendency of α^β [Bull. Acad. Sci. URSS Cl. Sci. Math. Nat. [Izvestiya Akad. Nauk SSSR] **1934**, 623–634]. *K. Mahler* (Manchester).

Postnikov, A. G. On the structure of two-dimensional Diophantine approximations. Doklady Akad. Nauk SSSR (N.S.) **76**, 493–496 (1951). (Russian)

Let θ be real and p_i, q_i be integer solutions of

$$|\theta-p_i/q_i|<c/q_i^2,$$

where $0<q=q_1<q_2<\cdots<q_t=Q$. Then $Q/q>((t-1)/2c)^{1/2}$, $Q/q>c^{-\frac{1}{2}[\frac{1}{2}(t-1)]}$. This improves a lemma of Khintchine. [Rend. Circ. Mat. Palermo **50**, 170–195 (1926)]. The following two-dimensional analogue is proved. Let $\theta_1>0$, $\theta_2>0$, and let $q_i>0$, $p_i>0$, r_i be N integer solutions of $|q_i\theta_1+p_i\theta_2-r_i|<c/q_ip_i$. Put $P=\max p_i$, $p=\min p_i$, $Q=\max q_i$, $q=\min q_i$. Then

$$(QP/qp)^2>(n+2m-2)/2nc,$$

where $n+m=N$ and n, m are respectively the number of points of the type $(q_i/r_i, p_i/r_i)$ on and inside their convex cover. *J. W. S. Cassels* (Cambridge, England).

ANALYSIS

***Fan, Ky. Les fonctions définies-positives et les fonctions complètement monotones. Leurs applications au calcul des probabilités et à la théorie des espaces distanciés.** Mémor. Sci. Math., no. 114. Gauthier-Villars, Paris, 1950. 48 pp. 400 francs

This booklet gives a survey of the various theories and problems in which completely monotone functions and Fourier-Stieltjes transforms of distributions occur. It is intended as an introduction for nonspecialists, and as such is easily readable. More technical proofs are omitted but detailed references are given. The first part is devoted to positive definite functions. Following Bochner they are identified with characteristic functions, and their occurrence in probability is indicated. More out of the usual path is the generalization introduced by Schoenberg to characterize point sets in Hilbert space. The second part is devoted to completely monotone functions, their characterization as Laplace integrals, interpolation problems, etc. Their occurrence in a special type of stochastic process is described. The most interesting section is again devoted to a survey of Schoenberg's investigations on isometric transformations of spaces and of completely monotone functions in Hilbert space.

W. Feller (Princeton, N. J.).

Mandelbrojt, Szolem. Théorèmes généraux de fermeture. C. R. Acad. Sci. Paris 232, 284–286 (1951).

Let $f(x)$, $g(x)\epsilon L(-\infty, \infty)$, and let $f(x)$ have p derivatives, also in L, $0\leqq p\leqq \infty$. Let $\Omega(f)$ be the set of points where the Fourier transform of f vanishes. Let E be an open set on the real axis R of the ζ-plane. If $E=\phi$, $\Delta_E=(\zeta|\eta>0)$, otherwise is the domain obtained by deleting $R-E\equiv CE$ from the ζ-plane. A positive function $M(r)$, $r>0$, is said to be associated with the set E if every function $\Phi(\zeta)$ holomorphic in Δ_E and satisfying $|\Phi(\zeta)|\leqq M(|\zeta|)|\eta|^{-1}$ is identically zero. The author is concerned with the problem of deciding when for every a the function $f(x+a)$ is the limit in L of finite combinations of the form $\sum a_n f^{(n)}(x)+\sum b_n g(x+\xi_n)$. Two solutions are given. For the first, one forms $M(r)=\inf M_n r^{-n}$, $0\leqq n<p+1$, $M_n=\|f^{(n)}\|$. If $M(r)$ is associated, for instance, with the set $C\Omega(g)$ or with $C\Omega(g)\cup \operatorname{int}\Omega(f)$, then the approximation theorem holds. This result contains various older ones as, for instance, Wiener's Tauberian theorem. For the second solution, take $p=\infty$, set $C(\sigma)=\sup(n\sigma-\log M_n)$, $n\geqq 0$, set $E=C\Omega(g)\cup \operatorname{int}\Omega(f)$ and let $\varphi(x)$ be the characteristic function of the set $R-(CE)_h$,

$$(A)_h=(x|y-h\leqq x\leqq y+h,\ y\epsilon A).$$

Suppose there exists a continuous function $u(x)$, $-\infty<x<\infty$, such that (1) $A\leqq u(x)\leqq \frac{1}{2}+\varphi(x)+B\varphi(x)\varphi(-x)$, A, B positive constants, (2) $u(x)\epsilon BV[-\infty, \infty]$ or $u'(x)$ continuous, $xu'(x)\epsilon BV$, $x[u'(x)]^2\epsilon L$, and (3)

$$\int^{\infty} C(\sigma)\exp\{-\int^{\sigma}[u(e^{\tau})+u(-e^{\tau})]^{-1}d\tau\}d\sigma=\infty.$$

Then the approximation theorem holds. [Author's correction: On p. 285, lines 3 and 6 transpose "l'ensemble vide" and "la droite entière".] *E. Hille.*

Mandelbrojt, Szolem. Théorèmes d'approximation et problèmes des moments. C. R. Acad. Sci. Paris 232, 1054–1056 (1951).

This is a continuation of the investigation described in the preceding review. Let $f\epsilon L(-\infty, \infty)$ and have derivatives of all orders, also in L, with $\|f^{(n)}\|\leqq M_n$. Let $S(f)$ be the Fourier transform of f, let $\tau(f)$ be the set of limit functions in L of functions of the form $\sum a_n f(x+\xi_n)$. Let $F=S(f)$, $\Psi=S(\psi)$ with $\psi\epsilon\tau(f)$. Let E be a closed set on R and let a continuous bounded function $u(x)$ satisfy conditions (2) and (3) of the preceding review, replacing (1) by (1') $\inf u(x)>0$, $u(x)\leqq\frac{1}{2}$ if $x\epsilon(E)_h$, $u(x)\leqq\frac{3}{2}$ if $x\bar{\epsilon}(E)_h$ and $-x\epsilon(E)_h$. Then to every $\epsilon>0$ there is a polynomial $P(u)$ such that $|\Psi(u)-P(u)F(u)|<\epsilon$ in E. Suppose further that $F(u)$ is a positive even function such that $-\log F(u)$ is a convex function of $\log|u|$ and $\lim_{u\to\infty}u^n F(u)=0$ for each n. Suppose that $u(x)$ satisfies (1'), (2), and (3) with $C(\sigma)$ replaced by $-\log F(e^{\sigma})$. If now $H(u)$ is any element of $C[-\infty, \infty]$, vanishing at $\pm\infty$, then to every $\epsilon>0$ there is a polynomial $P(u)$ such that $|H(u)-F(u)P(u)|<\epsilon$ on R. From the last result the author derives a general uniqueness theorem for the moment problem. Let $\{m_n\}$ be a given set of reals, let $V(e)$ be a distribution function of sets whose spectrum lies in the closed set of reals E, and such that $\int_{-\infty}^{\infty}u^n dV=m_n$, $n\geqq 0$. Let $\Phi(u)=\sum u^{2n}(m_{2n})^{-1}$ if E contains a negative number; otherwise $\Phi(u)=\sum u^{2n}(m_n)^{-1}$. Suppose that $u(x)$ satisfies (1'), (2), and (3) with $C(\sigma)$ replaced by $\log\Phi(e^{\sigma})$. Then the solution of the moment problem is essentially unique. For special choices of $u(x)$ the author obtains the uniqueness theorems of Carleman for the Hamburger and the Stieltjes moment problems. *E. Hille.*

Hsu, L. C. An asymptotic expression for an integral involving a parameter. Acad. Sinica Science Record 2, 339–345 (1949).

The author shows that the asymptotic formula

$$\int_a^b \exp[f(x, \lambda)]dx\sim\exp[f(\xi, \lambda)]\cdot\{-2\pi/f_{xx}(\xi, \lambda)\}^{\frac{1}{2}}$$

as $\lambda\to\infty$ holds under the following assumptions. The function $f(x, y)$ is real-valued, and continuous together with $f_x(x, y)$ and $f_{xx}(x, y)$ in the region $R(a\leqq x\leqq b\leqq\infty: N<y<\infty)$, and such that (i) $f_x(x, y)=0$ for each y and a suitable x in R and $f_{xx}(x, y)<0$ throughout R; (ii) there is a constant ξ $(a<\xi<b)$ such that $f_x(\xi, y)\to 0$ as $y\to\infty$; (iii) $f_{xx}(x, y)\to-\infty$ and $f_{xx}(x, y)/f_{xx}(\xi, y)\to 1$ as $(x, y)\to(\xi, \infty)$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Ostrowski, Alexandre. Un nouveau théorème d'existence pour les systèmes d'équations. C. R. Acad. Sci. Paris 232, 786–788 (1951).

In an earlier paper [same C. R. 231, 1114–1116 (1950); these Rev. 12, 445] the author obtained an existence theorem for a system of equations $f_\nu(x_1, \cdots, x_n)=0$ subject to certain inequalities. In the present note, the norms used are replaced by certain quantities which in the usual cases reduce to the maximum characteristic value of the Jacobian matrix for the system. *F. J. Murray.*

Ascoli, Guido. Sulle matrici permutabili con la propria derivata. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 245–250 (1950).

Let $X(t)$ be a matrix of order n whose elements are real or complex functions of the parameter t defined in the closed interval $\langle a, b\rangle$, such that its derivative $X'(t)$ exists and is finite in this interval. It is proved that if $X(t)$ is nonderogatory and commutes with $X'(t)$ in $\langle a, b\rangle$, then $X(t_1)$ and $X(t_2)$ commute for any t_1 and t_2 in $\langle a, b\rangle$. Further, $X(t)=\sum_1^n\mu_i(t)A_i$ where A_i are mutually commuting constant matrices and the functions $\mu_i(t)$ have finite derivatives in $\langle a, b\rangle$. An example shows that this result need not be true when $X(t)$ is derogatory in $\langle a, b\rangle$. *D. E. Rutherford* (St. Andrews).

Theory of Sets, Theory of Functions of Real Variables

Witt, Ernst. Beweisstudien zum Satz von M. Zorn. Math. Nachr. 4, 434–438 (1951).

A discussion of a maximality principle usually attributed to Zorn and a recent variant proposed by Kneser [Math. Z. 53, 110–113 (1950); these Rev. 12, 323]. The principle in question seems to have been first stated in general form by Hausdorff [Grundzüge der Mengenlehre, Veit, Leipzig, 1914, p. 140]. *A. D. Wallace* (New Orleans, La.).

Kaluza, Theodor, jr. Zu einer Wachstumsfrage bei Zuordnungen zwischen Ordinalzahlen. Math. Ann. 122, 323–325 (1950).

Let λ be a transfinite limit ordinal number, and call the set of ordinals α (well-ordered according to magnitude) such that $0<\alpha<\lambda$ the interval $(0, \lambda)$. To every $\alpha\epsilon(0, \lambda)$ let there correspond an ordinal $\varphi(\alpha)<\alpha$. The resulting function $\varphi(\alpha)$ is said to be definitely divergent in $(0, \lambda)$ provided that, given any $\beta_0<\lambda$, there exists an $\alpha_0\epsilon(0, \lambda)$ such that $\beta_0\leqq\varphi(\alpha)$ for every $\alpha\geqq\alpha_0$ in the interval $(0, \lambda)$. The author states a theorem of Dushnik [Bull. Amer. Math. Soc. 37, 860–862 (1931)] in the following equivalent form: If ω_δ is an initial number and δ is an isolated ordinal, then no definitely

divergent function exists in the interval $(0, \omega_1)$. He then shows that the question of the existence of a definitely divergent function in an interval $(0, \lambda)$ can be reduced, in certain cases, to the same question for certain intervals $(0, \lambda')$, where λ' is a transfinite limit ordinal less than λ. Finally, from these results it follows that (A) $\lambda=\omega_1$ is the smallest limit number such that no definitely divergent function exists in $(0, \lambda)$, and (B) $\lambda=\omega_{\omega_1}$ is the smallest initial number with limit index such that no definitely divergent function exists in $(0, \lambda)$. *F. Bagemihl.*

Helson, Henry. On a problem of Sikorski. Colloquium Math. **2**, 7–8 (1949).

For every $\alpha<\omega_\mu$, let M_α denote a set of sequences of type α composed of zeros and ones, and suppose that, for every pair of ordinal numbers α and β with $\alpha<\beta<\omega_\mu$, M_α is the set of segments of type α of the sequences of M_β. The problem in question is: Is there necessarily a set M of sequences of type ω_μ such that every M_α, $\alpha<\omega_\mu$, is the set of segments of type α of the sequences of M? Let $\gamma=cf(\omega_\mu)$. The answer is affirmative if $\gamma=0$, negative if γ is an isolated number, and the problem is open if γ is a transfinite limit number. *F. Bagemihl* (Rochester, N. Y.).

Specker, E. Sur un problème de Sikorski. Colloquium Math. **2**, 9–12 (1949).

The problem is the same as in the paper reviewed above, with the additional hypothesis that the cardinal number of M_α is less than $\aleph_\mu$ for every $\alpha<\omega_\mu$. Inspired by Helson's method, the author shows that the answer is negative for $\mu=1$. Sikorski and the author observe that, under the generalized hypothesis of the continuum, the answer is negative for $\mu=\nu+1$ if ω_ν is a regular initial number. *F. Bagemihl* (Rochester, N. Y.).

Iseki, Kiyoshi. On singular sets. II. On the S. Piccard's theorem. J. Osaka Inst. Sci. Tech. Part I. **1**, 75 (1949).

S. Piccard [Sur les ensembles de distances des ensembles de points d'un espace Euclidien, Neuchâtel, 1939; these Rev. **2**, 129] proved the theorem: The system of distance sets of the point sets in a Euclidean space has the power $2^{\aleph_0}$ [loc. cit., p. 16]. The author gives a new proof of this theorem by means of a Hamel basis B. [Reviewer's remark: The relation $B\cap B(a)=0$ for $a\neq 0$, used by the author, is not correct (take $a=x'-x$ with x and $x'\epsilon B$). But this relation can be replaced by the fact that $x'=x\pm(x''-x''')$, with $x''\neq x\neq x'''$ and $x, x', x'', x'''\epsilon B$, is impossible.] *A. Rosenthal* (Lafayette, Ind.).

Sierpiński, Wacław. Sur un ensemble plan singulier. Fund. Math. **37**, 1–4 (1950).

If A and B are plane sets, then $A\simeq B$ means that A can be superimposed on B by a translation or rotation. The following theorems are proved: (I) Every subset E of the line contains at most one point p such that $E-p\simeq E$. (II) There is a set E in the plane containing two different points p, q, such that $E-p\simeq E\simeq E-q$. Theorem (II) is proved by giving an explicit definition of such an E. *E. E. Moise* (Ann Arbor, Mich.).

Ščegol'kov, E. A. Elements of the theory of B-sets. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 14–44 (1950). (Russian)

This memoir is a clearly written survey of the descriptive theory of Borel sets in the space of irrational numbers. Among the topics treated are the following: the classification of Luzin-de La Vallée Poussin; the theory of separability; the classification of Hausdorff; the theorem of Aleksandrov-Hausdorff that every uncountable Borel set (in this space) has cardinal number $2^{\aleph_0}$. *E. Hewitt.*

Arsenin, V. Ya., and Lyapunov, A. A. The theory of A-sets. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 45–108 (1950). (Russian)

The authors present a detailed survey of the theory of analytic sets in the space of irrational numbers. Nothing of a novel character is included, but the memoir is to be highly commended for completeness and lucidity. Frequent reference is made to the paper of Ščegol'kov reviewed above. *E. Hewitt* (Uppsala).

Očan, Yu. S. The equivalence of families of B-sets. Uspehi Matem. Nauk (N.S.) **5**, no. 6(40), 139–142 (1950). (Russian)

Two families $\{E_\alpha\}$ and $\{E'_\beta\}$ of subsets of a set R are said to be equivalent if there exists a one-to-one mapping of R into R such that the first family is mapped exactly onto the second family. Szpilrajn [Fund. Math. **26**, 302–326 (1936)] has proved the following result: Every countable family of B-sets in the space of irrational numbers is equivalent to a countable family of open-and-closed sets of this space, if a certain countable set of points is removed. Extensions of and limitations on this result for uncountable families of sets are discussed in the present paper. Let R be a complete metric space with a countable basis. A family $\mathfrak{N}$ of subsets of R is said to be an N-family if (1) every open subset of R can be obtained from sets in $\mathfrak{N}$ by iterations of the formation of countable unions and countable intersections, (2) every perfect compact subset of R is contained in some set of the family $\mathfrak{N}$. The basic theorem proved is: An N-family of subsets of R, every element of which possesses a certain topologically invariant property K, is not equivalent to any family $\mathfrak{M}$ of subsets of R which consists entirely of B-sets which do not possess the property K. As corollaries of this result, one obtains: The family of perfect compact sets in R is equivalent to no family of B-sets none of which is perfect and compact; for $\zeta\geqq 2$, the family of all G^ζ sets (F^ζ sets) is not equivalent to any family of sets which consists entirely of F^ζ sets (G^ζ sets). Finally, it is shown that the families of B-sets, A-sets, and CA-sets all have the property that they are equivalent to no proper subfamilies of themselves. *E. Hewitt* (Uppsala).

Lyapunov, A. A. On the equivalence of families of sets. Uspehi Matem. Nauk (N.S.) **5**, no. 6(40), 143–144 (1950). (Russian)

[For terminology and notation, see the preceding review.] It is proved that there exists no family $\mathfrak{M}$ of subsets of R consisting of $\mathfrak{c}$ elements such that every family containing $\mathfrak{c}$ sets is equivalent to some subfamily of $\mathfrak{M}$. *E. Hewitt* (Uppsala).

Lyapunov, A. A. B-functions. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 109–119 (1950). (Russian)

Let X be a set and let Y be a topological space. Consider an initial class of functions mapping X into Y, which may be designated either as the 0 or the 1 class of functions. Let α be an ordinal number less than Ω, and suppose that for all $\beta<\alpha$, functions of class β have been defined. Then functions of class α are all pointwise limits of convergent se-

quences of functions of class less than α. These are Baire functions, and the classes of functions are Baire classes. For $H \subset Y$ let $[f(x) \in H]$ denote the set of points $x \in X$ such that $f(x) \in H$. As H runs through the families of all closed sets and all open sets in Y, the sets $[f(x) \in H]$ describe the Lebesgue sets of the function f. If X is a topological space, Borel sets in X are defined as follows. Let $X = \bigcup_n E_n$, where the sets E_n are open or closed and are pairwise disjoint. All sets obtainable as unions of subsequences of such sequences $\{E_n\}$ are sets of the 1st class. If $\alpha < \Omega$ and all classes of sets with number less than α have been defined, then the class of sets E obtainable in the form $E = \lim_{n\to\infty} E_n$, with E_n of class less than α, is the class of sets α. The union of all these classes is the class of B-sets. [Reviewer's note: Observe the difference between this definition and the usual definition of Borel sets.] A function all of whose Lebesgue sets are B-sets is a B-function. A function $y = f(x)$ such that the set of all $(x, f(x))$ in $X \times Y$ is a B-set in $X \times Y$ is said to be a quasi-B-function.

For the case $X = Y =$ the space of irrational numbers, one takes the class of continuous functions as the initial class of functions, and Lebesgue sets for open-and-closed sets as the sets of class 0. Then the following assertions are true: (I) (Lebesgue) A function $y = f(x)$ is of class α if and only if all of its Lebesgue sets for open-and-closed sets are of class α; (II) (Lebesgue) the class of B-functions is identical with the class of Baire functions; (III) (Luzin) the class of B-functions is identical with the class of quasi-B-functions.

Suppose that X and Y are complete metric spaces with countable bases. Then one may take the functions of class 1 to be the uniform limits of pointwise limits of continuous functions. In this case, a function is of class α if and only if its Lebesgue sets for closed sets are countable intersections of sets of class α. Theorems (II) and (III) above are true without alteration.

It is next shown that separable real Hilbert space admits a one-to-one mapping onto the space of irrational numbers such that the image and inverse image of a B-set of class $\alpha > \omega$ is again a set of the same class. Observations are next made concerning functions $\varphi(x, y) = 0$, where $\varphi(x, y)$ is a real B-function defined on $X \times Y$ (X and Y are complete metric spaces with countable bases). The set of all $x \in X$ for which there is a y such that $\varphi(x, y) = 0$ is always an A-set, and if the set of all such y is countable for every x, then the set of x is a B-set. If the implicit function $\varphi(x, y) = 0$ is single-valued, then it is a B-function. Finally, the author shows that every B-function on a complete metric space is continuous on a G_δ of the 2d category. *E. Hewitt.*

Mackina, R. Yu. On continuous images of Hilbert space. Izvestiya Akad. Nauk SSSR. Ser. Mat 15, 95–103 (1951). (Russian)

Let H and H^* be separable real Hilbert spaces. Then, for every $\alpha < \Omega$, there exists a continuous mapping of H into H^* such that the image of H is a B-set strictly of class α. There also exists a continuous image of H in H^* which is an A-set and not a B-set. (This result can be achieved even by mapping H into the Euclidean plane.) Finally, there exists a one-to-one continuous image of H in H^* which is a B-set of arbitrarily high order. The constructions employed to obtain these results use only the facts that H contains a closed set homeomorphic to the space of irrational numbers and that H is a complete separable metric space. Thus the results described can be automatically extended to any such space. *E. Hewitt* (Uppsala).

Dubrovskiĭ, V. M. On the property of equicontinuity of a family of completely additive set functions with respect to proper and improper bases. Doklady Akad. Nauk SSSR (N.S.) 76, 333–336 (1951). (Russian)

The author produces yet another extension of his work on families of measures [same Doklady (N.S.) **58**, 737–740 (1947); **63**, 483–486 (1948); these Rev. **9**, 275; **10**, 361]. For terminology and notation, see the first review cited. A basis M for a family $\{\Phi_\lambda\}_{\lambda \in A}$ of set-functions is said to be proper if $M(E)$ is finite for all $E \in \mathfrak{M}$, and improper if M is σ-finite but assumes infinite values. The author proves that if $\{\Phi_\lambda\}_{\lambda \in A}$ is equi-continuous with respect to some proper basis, then it is equi-continuous with respect to all bases, proper or improper. He also shows that equi-continuity with respect to an improper basis does not imply either uniform additivity or equi-continuity with respect to a proper basis. Finally, he establishes an elementary and well-known result on absolute continuity [found, for example, in Halmos, Measure Theory, Van Nostrand, New York, 1950, p. 125, theorem B; these Rev. **11**, 504]. *E. Hewitt.*

Berger, Agnes. Remark on separable spaces of probability measures. Ann. Math. Statistics 22, 119–120 (1951).

If a set M of probability measures with a common domain is topologized by using as a subbase at m_0 the class of all sets of the form $\{m: |m(B) - m_0(B)| < \epsilon\}$, and if in the topological space so obtained there is a countable dense set, then there exists a measure $\bar{m}$ such that $\bar{m}(B) = 0$ implies that $m(B) = 0$ for all m in M. *P. R. Halmos.*

★Natanson, I. P. Teoriya funkciĭ veščestvennoĭ peremennoĭ. [Theory of Functions of a Real Variable]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 399 pp.

This book would be an excellent text for a first-year course in real-variable theory for such students as can read Russian. The discussion of the usual topics is clear and well arranged. Chapters (I)–(IX), emphasizing the case of one variable, discuss infinite sets, point sets, measurable sets and functions, Lebesgue integration, L^p spaces, bounded variation and Stieltjes integrals, and absolute continuity and differentiability. The remaining chapters extend and apply these basic results; the chapter headings are: (X) Singular integrals, trigonometrical series. (XI) Point sets in two-dimensional spaces. (XII) Measurable functions of several variables. (XIII) Set functions and their application to the theory of integration. (XIV) Transfinite numbers. (XV) The classification of Baire. (XVI) Some concepts of functional analysis. Chapter (XVII) is a historical survey of the role of Russian and Soviet scholars in extending the theory of functions of a real variable. As an illustration of the lengths to which a profitable topic is pursued, the invariance of inner and outer measure under Euclidean motions is demonstrated early; it is applied to the construction of a nonmeasurable set in R_1. Later the author shows that this example shows the nonexistence of a completely additive measure invariant under all motions of R_n and defined on all subsets of R_n, $n \geqq 1$. He quotes Banach's and Hausdorff's results on the existence for $n \leqq 2$ and nonexistence for $n > 2$ of a finitely additive, invariant measure. Hausdorff's result is proved in chapter (XI) by displaying his "paradoxical" decomposition of the sphere.

M. M. Day (Urbana, Ill.).

Love, E. R. A generalization of absolute continuity. J. London Math. Soc. **26**, 1–13 (1951).

In this paper the author defines for $p \geqq 1$ the class V^p of pth power absolutely continuous functions $f(x)$ to be those which satisfy on a finite closed interval $[a, b]$ the inequality $\sum |f(\beta_\nu)-f(\alpha_\nu)|^p<\epsilon$ for all finite sets of nonoverlapping intervals (α_ν, β_ν) such that $\sum(\beta_\nu-\alpha_\nu)^p<\delta=\delta(\epsilon)$. This obviously reduces to the ordinary class of absolutely continuous functions for $p=1$, and the author shows that for $p>1$ it reduces to the continuous members of a class of functions studied earlier by L. C. Young and the author. The relationship between this concept and the pth power total variation

$$V_p(f; a, b)=\sup_{a=x_0<x_1<\cdots<x_k}\left(\sum_{i=1}^{k}|f(x_i)-f(x_{i-1})|^p\right)^{1/p}$$

defined by Wiener [J. Math. Phys. **3**, 73–94 (1924)] is studied, and in particular it is proved that for $p \geqq 1$, a necessary and sufficient condition that $f(x) \in V^p(a, b)$ is that $f(x)$ be measurable on $[a, b]$ and

$$V_p\{f(x+h)-f(x); a \leqq x \leqq b-h\} \rightarrow 0$$

as $h \rightarrow 0^+$. For $p=1$, this reduces to known results of Wiener, R. C. Young, and Ursell. *R. H. Cameron.*

Love, E. R. More-than-uniform almost periodicity. J. London Math. Soc. **26**, 14–25 (1951).

In this paper the author defines variation almost periodic (a.p.) functions in terms of the norm

$$D_{V^p}(f)=\sup_{-\infty \leqq x \leqq \infty}|f(x)|+\sup_{-\infty \leqq x \leqq \infty} V_p(f; x, x+1).$$

(Here $V_p(f; a, b)$ and the class V^p are defined as in the paper reviewed above.) Specifically, $f(x)$ is V^p-a.p. if it belongs to V^p on every finite interval and possesses for each $\epsilon>0$ a relatively dense set of translation numbers t such that $D_{V^p}\{f(x+t)-f(x)\} \leqq \epsilon$. It is shown (among other things) that a necessary and sufficient condition that a function be V^1-a.p. is that it be the indefinite integral of a Stepanoff a.p. function and be bounded. It is shown that the space of V^p-a.p. functions is the D_{V^p} closure of the space of trigonometric polynomials. *R. H. Cameron.*

Kozlov, V. Ya. Gol'dovskiĭ's example. Mat. Sbornik N.S. 28(70), 197–204 (1951). (Russian)

Following an idea given by Gol'dovskiĭ in a lecture in 1930 (but not published owing to his death in 1931), the author constructs an example of a function $f(x)$, continuous on the closed interval $[0, 1]$, such that its exact derivative $f'(x)$ exists ($+\infty$ or $-\infty$ allowed) at every point of $[0, 1]$ but is not integrable over this interval in the (restricted) sense of Denjoy. The Lebesgue integral of $f'(x)$ over $[0, 1-\epsilon]$ exists, if $0<\epsilon \leqq 1$, but does not converge as $\epsilon \rightarrow 0$. The author gives also a second example in which Denjoy's construction breaks down at the next stage, i.e., owing to nonconvergence of the sum of contributions from the complementary intervals of the set of points of nonsummability.

H. P. Mulholland (Bath).

Sargent, W. L. C. On the continuity (C) and integrability (CP) of fractional integrals. Proc. London Math. Soc. (2) **52**, 253–270 (1951).

The mean of order α on $[c, d]$ is, for suitable $f(t)$,

$$C_\alpha(f, c, d)=\int_c^d |d-t|^{\alpha-1} f(t)dt \Big/ \int_c^d |d-t|^{\alpha-1}dt.$$

If $C_\alpha(x, x+h)$ is bounded for all sufficiently small h then $f(x)$ is bounded (C, α) at the point x. If $C_\alpha(x, x+h) \rightarrow f(x)$ as $h \rightarrow 0$ then $f(x)$ is C_α-continuous at x. The C_α-derivatives are defined as the limits of $[C_\alpha(f, x, x+h)-f(x)]/\{h/(\alpha+1)\}$; $\omega_\alpha(f, c, d)=$

$$\max\{\overline{\text{bound}}_{c<t<d}|C_\alpha(f, c, t)-f(c)|, \overline{\text{bound}}_{c<t<d}|C_\alpha(f, d, t)-f(d)|\}.$$

If $f(x)$ is absolutely continuous over a bounded closed set Q, if $f(t)$ is integrable $C_\mu P$ on each interval (c_r, d_r) of the set complementary to Q, $\mu=\max(\lambda-1, 0)$, if $\sum \omega_\lambda(f, c_r, d_r)$ converges, then $f(t)$ is AC^* (C_λ-sense) over Q. If $[a, b]$ is covered by a sequence $Q_1, Q_2, \cdots$ on each set of which $f(t)$ is AC^*, then $f(t)$ is ACG^* on $[a, b]$.

If $f(x)$ is measurable and bounded then the Riemann-Liouville fractional integral

$$F_\alpha(x)=[\Gamma(\alpha)]^{-1}\int_a^x (x-t)^{\alpha-1}f(t)dt$$

is continuous in (a, b) if $\alpha>0$. This result has been extended by Hardy and Littlewood to the cases when $f(x)$ satisfies a Lipschitz condition and when $f(x)$ is not necessarily bounded but belongs to some L^p, $p \geqq 1$. The paper under review extends these results to the case when $f(x)$ is not necessarily absolutely integrable, but is bounded (C, λ), $\lambda \geqq 0$. If $0 \leqq \alpha \leqq \lambda$, then $F_\alpha(x)$ is C_γ-continuous whenever $\gamma>\lambda-\alpha$, but not in general $C_{\lambda-\alpha}$-continuous. If $0 \leqq \lambda<\alpha<\lambda+1$, $F_\alpha(x)$ belongs to Lip $(\alpha-\lambda)$. If $\lambda=n+p$, n zero or a positive integer, $0<p<1$, there is $f(x)$ which is ACG^* (C_λ-sense) on $[a, b]$ and such that $F_\lambda(x)$ is not bounded on $[a, b]$. If $0<\alpha \leqq \lambda$, $\gamma>\lambda-\alpha$, $f(x)$ bounded (C, λ) on $[a, b]$, then $F_\alpha(x)$ is C_γ-continuous on $[a, b]$. If $\alpha \geqq 0$ and $f(x)$ is $C_\gamma P$-integrable then $F_{\alpha+1}(x)$ is the $C_\gamma P$-integral of $F_\alpha(t)$ on $a \leqq t \leqq x$, $a \leqq x \leqq b$, where $\gamma>\lambda-\alpha$, $0 \leqq \alpha \leqq \lambda$, and $\gamma=0$ if $\alpha>\lambda$.

R. L. Jeffery (Kingston, Ont.).

Hammersley, J. M. A theorem on multiple integrals. Proc. Cambridge Philos. Soc. **47**, 274–278 (1951).

This paper contains a discussion of the general theorem on the behavior of integrals under transformations, and several corollaries and special cases of that theorem. (The theorem asserts that, under suitable hypotheses, $\int f(Tx)d\mu(x)=\int f(x)d\mu(T^{-1}x)$.) The author seems not to be aware that the results are known [cf., e.g., the reviewer's book, Measure Theory, Van Nostrand, New York, 1950, §39; these Rev. **11**, 504]. *P. R. Halmos* (Chicago, Ill.).

Popoff, Kyrille. Sur une extension de la notion de dérivée au moyen de la théorie des probabilités. Časopis Pěst. Mat. Fys. **74** (1949), 220–229 (1950). (French. Czech summary)

Consider a plane curve C passing through the point $P(x_0, y_0)$ and let $L(x)$ be the line $y-y_0=(x-x_0)\tan\alpha$. Consider the mean square distance of C from $L(\alpha)$ over an interval $x_0<x<x_0+h$ and the two values of α for which it assumes an extremum. The limiting positions of the corresponding lines are defined as generalized tangent and normal of C at P, and in this way one has a generalized notion of derivative of a function $f(x)$. This notion is extended to functions of several variables and to complex functions. Applications to convex surfaces are made following Busemann and Feller [Acta Math. **66**, 1–47 (1936)]. [Probability does not enter the theory unless a quadratic mean is considered its monopoly.]

W. Feller (Princeton, N. J.).

Theory of Functions of Complex Variables

Denjoy, Arnaud. Une expression de la fonction $\zeta(s)$ de Riemann. C. R. Acad. Sci. Paris **232**, 905–908 (1951).

The author develops the representation

$$\zeta(s)=2^s\lim_{p\to\infty}\int_0^\infty y^{-s}[\tfrac{1}{2}-\lambda_p(y)]dy$$

of the Riemann zeta-function in the region $0<\Re[s]<1$, where

$$\lambda_p(y)=\frac{\rho^p}{2\sin\frac{1}{2}y}\sin(p\theta-\tfrac{1}{2}y),$$

$$\rho\cos\theta+i\rho\sin\theta=1-\frac{\sin y}{y}+i\frac{1-\cos y}{y}.$$

P. R. Garabedian (Stanford University, Calif.).

Kemperman, J. H. B. Asymptotic expansion of entire functions defined by power series. Math. Centrum Amsterdam. Rapport **ZW-1950-018**, 17 pp. (1950). (Dutch)

The problem is to express the asymptotic behavior of $\sum_{n=0}^\infty g(an+b)z^n$ in terms of that of $\sum_{n=0}^\infty g(n)z^n$. The author generalizes the results of Wright [Philos. Trans. Roy. Soc. London. Ser. A. **238**, 423–451 (1940); these Rev. **1**, 212] by weakening the restrictions on the function $g(z)$ and enlarging the region in which the asymptotic expansion is shown to hold. The detailed statements are too long to quote here.

R. P. Boas, Jr. (Evanston, Ill.).

Cowling, V. F. On the analytic continuation of Newton series. Proc. Amer. Math. Soc. **2**, 28–31 (1951).

The author shows that if the function

$$f(z)=\sum_{n=0}^\infty a(n)(-1)^n C_{z-1,n}$$

has an abscissa of convergence $\sigma<\infty$, then $f(z)$ is entire, where $a(w)$ is regular in the region $|\arg w-h|\leqq B$ for $h>0$ and $0<B<2\pi$, and where $|a(h+Re^{i\psi})|\leqq R^K\exp(-LR\sin\psi)$ for some K and some L, $0<L<2\pi$, and for all large R.

A. J. Lohwater (Ann Arbor, Mich).

Ilieff, Ljubomir. Über in der Umgebung der Abzisse der absoluten Konvergenz einer Klasse Dirichletscher Reihen zugehörige singuläre Stellen. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **43**, 239–267 (1947). (Bulgarian. German summary)

[Volume number misprinted 42 on title page]. Proofs are given of results announced elsewhere [C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 2–3, 19–22 (1948); these Rev. **11**, 168].

W. Seidel (Rochester, N. Y.).

Ilieff, Ljubomir. Über die Verteilung der Nullstellen einer Klasse ganzer Funktionen. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. (Math. Phys.) **44**, 143–174 (1948). (Bulgarian. German summary)

Let $f(t)$ be nonnegative and integrable in (0, 1) such that the zeros of the function $\int_0^1 f(t)\{p(z+t)-p(z-t)\}dt$, where $p(u)$ denotes a polynomial or entire function of u, lie in $\alpha\leqq\Re(z)\leqq\beta$. Let $\psi(t)$ be positive and integrable in (0, 1). Setting $x=\int_t^1\psi(t)dt$, $f(t)=\psi(t)f_0(t)$, $f_0(t)=\varphi_0(x)$, and

$$\int_0^x\varphi_0(x)dx=\varphi_1(x)=f_1(t),$$

the zeros of the function $\int_0^1 f_1(t)\{p'(z+t)+p'(z-t)\}dt$ lie also in $\alpha\leqq\Re(z)\leqq\beta$. Special cases of this theorem are treated separately. This includes other results of the author [C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 2–3, 9–11 (1949); these Rev. **12**, 15].

W. Seidel (Rochester, N. Y.).

Noble, M. E. Extensions and applications of a Tauberian theorem due to Valiron. Proc. Cambridge Philos. Soc. **47**, 22–37 (1951).

As counterparts to classical theorems concerning entire functions whose zeros lie on a half-line, the author proves theorems on functions whose zeros lie on or near a whole line, the object being to deduce the distribution of the zeros from the growth of the function along the positive real axis. The main theorem is that, if $f(z)$ is of order σ, $0<\sigma<1$, with zeros a_n such that $\Re(a_n)=o(|a_n|)$, and with $\log|f(x)|\sim\frac{1}{2}\pi\lambda x^\rho\csc\frac{1}{2}\pi\rho$ as $x\to\infty$, $0<\rho<1$, then $n(r)\sim\lambda r^\rho$. The proof is rather difficult, one of the chief difficulties being to show that $f(z)$ is actually of order ρ, mean type. Some additional results are given for functions of order 1, for example the following. Suppose that $f(z)$ has order 1, purely imaginary zeros, and a canonical product of genus 1. Then $n(r)\sim2\pi^{-1}Kr\log r$ if $\log|f(x)|\sim Kx\log x$. Further results are given for zeros which are only near the imaginary axis and for functions of finite type.

R. P. Boas, Jr.

Popken, J. An arithmetical theorem concerning linear differential equations of infinite order. Nederl. Akad. Wetensch., Proc. **53**, 1645–1656 = Indagationes Math. **12**, 522–533 (1950).

The number ζ is called an exceptional point of the integral function $y(z)$ if ζ and $y(\zeta)$ are both algebraic numbers; and if ζ, $y^{(j)}(\zeta)$, $j=0, 1, \cdots, \mu-1$ are algebraic but $y^{(\mu)}(\zeta)$ is not, then ζ is an exceptional point of multiplicity μ ($\mu=\infty$ if ζ and all derivatives at $z=\zeta$ are algebraic). The principal result is theorem 1: Let the integral function $y(z)=\sum_0^\infty c_nz^n/n!$, with $\limsup|c_n|^{1/n}\leqq q$ $(q>0)$, satisfy the linear differential equation of infinite order $\sum_0^\infty a_ny^{(n)}(z)=0$, where $\{a_n\}$ are constants not all zero. Let the function $F(t)=\sum_0^\infty a_nt^n$ be regular in $|t|\leqq q$ and let ν be the maximum of the multiplicities of the zeros of $F(t)$ in $0<|t|\leqq q$. Then: (i) If $y(z)$ has ν or more exceptional points ζ $(\neq0)$, where an exceptional point of multiplicity μ is counted μ times, then $y(z)$ is a polynomial with algebraic coefficients. (ii) If $a_0\neq0$, and $\zeta\neq0$ is an exceptional point with multiplicity μ, then ζ is a zero of $y(z)$ of order μ. A corollary is theorem 2: Let the entire transcendental function $y(z)=\sum_0^\infty c_nz^n/n!$ with $c_n=O(q^n)$ and c_n algebraic, satisfy the linear differential-difference equation $\sum_{\mu=0}^m\sum_{\nu=0}^n A_{\mu\nu}y^{(\mu)}(z+\omega_\nu)=0$, where the $A_{\mu\nu}$ are constants not all zero, and the constants ω_ν are distinct. Then $y(z)$ is a transcendental number for all but a finite number of algebraic values of z. Moreover, if $\sum_{\nu=0}^n A_{0\nu}\neq0$, then each exceptional point ζ $(\neq0)$ is a zero of $y(z)$.

I. M. Sheffer (State College, Pa.).

Bazilevič, I. E. On distortion theorems and coefficients of univalent functions. Mat. Sbornik N.S. **28(70)**, 147–164 (1951). (Russian)

By integrating a differential equation of Loewner type for functions of the class Σ_2: $F(\zeta)=\zeta+\alpha_1\zeta^{-1}+\alpha_3\zeta^{-3}+\cdots$ univalent and odd for $|\zeta|>1$, it is proved that if $F(\zeta)\varepsilon\Sigma_2$, then for any two points in $|\zeta|>1$,

$$\text{(A)}\quad\left|\log\left\{\frac{F(\zeta_1)+F(\zeta_2)}{F(\zeta_1)-F(\zeta_2)}\cdot\frac{\zeta_1-\zeta_2}{\zeta_1+\zeta_2}\right\}\right|^2 \leqq\log\frac{|\zeta_1|^2+1}{|\zeta_1|^2-1}\log\frac{|\zeta_2|^2+1}{|\zeta_2|^2-1}.$$

Various results are deduced from (A) by specializing the points selected so that $|\zeta_1|=|\zeta_2|$ and $|F(\zeta_1)|=|F(\zeta_2)|$, and again by taking real and imaginary parts. These latter inequalities are sharp, equality occurring for $F(\zeta)=\zeta\pm\zeta^{-1}e^{-2i\alpha}$. Let S_s denote the class of functions $f(z)=z+\cdots$ which are univalent, regular, and have s-fold symmetry in $|z|<1$, $s=1, 2, \cdots$. Setting $f(z)=F^{-1}(z^{-1})$, an inequality similar to (A) is derived for functions of the class S_2. Specialization yields the result that if $f(re^{i\varphi})=Re^{i\psi}\varepsilon S_s$ and if z_1 and z_2 in $|z|<1$ are such that $r_1=r_2$ and $R_1=R_2$ then

$$\text{(B)}\quad \frac{1-r^s}{1+r^s}\left|\tan\frac{s(\varphi_1-\varphi_2)}{4}\right|\leqq\left|\tan\frac{s(\psi_1-\psi_2)}{4}\right| \leqq\frac{1+r^s}{1-r^s}\left|\tan\frac{s(\varphi_1-\varphi_2)}{4}\right|.$$

Let $l_s(r, x)$ denote the linear measure of the set common to the circle $|w|=x$ and the region $D_s(r)$ which is the image of $|z|\leqq r<1$ under $w=f_s(z)\varepsilon S_s$. Using (B) the sharp upper bound for $l_s(r, x)$ is determined for x in the range

$$re^{\pi/es}\leqq x\leqq r/(1-r^s)^{2/s},$$

equality occurring for $f_s(z)=z/(1-z^s)^{2/s}$. Further, an upper bound for the area of $D_s(r)$ is obtained, which is not quite sharp due to the limitations on x. The bound on the area of $D_s(r)$ is used to prove that if $f(z)=z+c_2z^2+\cdots\varepsilon S_1$, then $|c_n|<\frac{1}{2}ne+1.51$. *A. W. Goodman.*

Oğuztöreli, Namık. Sur une généralisation de la formule de Jensen et quelques applications. Rev. Fac. Sci. Univ. Istanbul (A) **15**, 289–332 (1950). (French. Turkish summary)

The well-known formula of Jensen, relating the moduli of the zeros and poles in $|z|<r$ of a function meromorphic in $|z|\leqq r$ to the mean value of $\log\left|\frac{f(z)}{f(0)}\right|$ on $|z|=r$, is generalized in this paper in several directions. Jensen's formula has been extended by Pfluger [Comment. Math. Helv. **13**, 284–292 (1941); these Rev. **3**, 202] to sectors of circular rings. The author first extends the formulae of Pfluger to simplify connected domains (η) with the following properties: (1) The origin is an exterior point of (η); (2) the boundary C of (η) is a Jordan rectifiable curve such that every ray from the origin and every circle $|z|=r$, when not a part of the boundary, cut the boundary in just two points (or not at all). Let R and r be the maximum and minimum moduli, and β and α the maximum and minimum arguments, of points on C. Let $f(z)$ be meromorphic in (η) with $n_0(\rho)$ and $n_\infty(\rho)$, respectively, the number of zeros and poles of $f(z)$ in the intersection of (η) by the circle $|z|<\rho$, $r\leqq\eta\leqq R$. Let $n_0(\theta)$ and $n_\infty(\theta)$ be respectively the number of zeros and poles of $f(z)$ in the intersection of (η) by the angle $\alpha\leqq\phi<\theta$ $(\theta\leqq\beta)$. Then the extended formula is

$$\text{(A)}\quad \frac{1}{2\pi i}\int_{(C)}\log z\frac{dz}{z}=\int_r^R\{n_0(\rho)-n_\infty(\rho)\}\frac{d\rho}{\rho} +i\int_\alpha^\beta\{n_0(\theta)-n_\infty(\theta)\}d\theta.$$

Several modifications of formula (A) are considered when (1) the origin is an interior point of (η), (2) for more general domains (η') made from the union of a finite number of domains (η), (3) the domains (η) are unbounded with the contour C possessing an asymptote. In the latter case sufficient conditions are given so that $w=f(z)$ has no zeros in (η). *M. S. Robertson* (New Brunswick, N. J.).

Sakai, Eiichi. On the multivalency of analytic functions. J. Math. Soc. Japan **2**, 105–113 (1950).

Seven theorems are enunciated in which the conclusion is always that $f(z)=c_nz^n+c_{n+1}z^{n+1}+\cdots$ is n-valent and star-shaped in a circle $|z|<\rho$. The hypotheses are $c_n\neq0$, $f(z)$ is regular in $|z|<1$, and, for example, either $c_n=1$, $|f(z)|<M$ [theorem 3, due to Loomis, Bull. Amer. Math. Soc. **46**, 496–501 (1940); these Rev. **1**, 308] or $\Re\{f(z)/z^n\}>0$ [theorem 4], $c_n=1$, $0<|f(z)|<M$ [theorem 6]. Exact values of ρ and the associated extremal functions are given for these three theorems. The other four are of a more general nature. The case $n=1$ generalises work of Dieudonné [Ann. Sci. École Norm Sup. (3) **48**, 247–272, 273–320, 321–358 (1931)]. *A. J. Macintyre* (Aberdeen).

Dugué, Daniel. Sur les valeurs exceptionnelles de fonctions ayant plusieurs singularités essentielles. C. R. Acad. Sci. Paris **232**, 380–381 (1951).

Following Maillet [J. Math. Pures Appl. (5) **8**, 329–386 (1902)], the author has defined quasi-meromorphic functions as having a finite number of essential points, not necessarily isolated, and otherwise regular except for poles. He considers a generalisation of Picard's theorem for such functions. Specifically, he takes functions with two essential points, at the origin and at infinity. Ten cases arise according as there are none, one, or two exceptional values near either essential point. Examples are given of nine of these and the tenth is shown to be impossible by the following theorem: If there are two exceptional values at each essential point then one at least must be the same for both. It is stated that the theory holds equally for values exceptional in the sense of Borel. An extension to the case of singular lines is suggested. *R. Wilson* (Swansea).

Finzi, Arrigo. Sulle trasformazioni conformi del piano e su due possibili estensioni del teorema di Cauchy. Ann. Scuola Norm. Super. Pisa (3) **4**, 191–203 (1950).

Let $z'=x'(x, y)+iy'(x, y)=f(z)$ denote an angle-preserving correspondence between the points of a region D of the z-plane and a subset of the z'-plane, $z=x+iy$, $z'=x'+iy'$. Let z_1 be an arbitrary point of D, and let $\lim_{z\to z_1}[f(z)-f(z_1)]/(z-z_1)$ exist for every direction through z_1. If (1) the modulus of this limit is independent of direction for all $z_1\varepsilon D$, or if (2) the argument of this limit is independent of direction for all $z_1\varepsilon D$, then $f(z)$ is analytic in D. The technique consists of showing that the validity of either (1) or (2) implies the validity of the other almost everywhere in D, so that the method used in Goursat's proof of Cauchy's theorem can be applied. *A. J. Lohwater.*

Mori, Akira. On a conformal mapping with certain boundary correspondences. J. Math. Soc. Japan **2**, 129–132 (1950).

The following theorem is established with the aid of a theorem of Evans [Monatsh. Math. Phys. **43**, 419–424 (1936)]: If E is a set on the unit circle whose closure is of logarithmic capacity zero, there exists a function w which satisfies the following conditions: (1) it maps the interior of the unit circle one-to-one directly conformally onto a region D which consists of the plane cut along a countable set of radial slits clustering only at infinity; (2) each point of E corresponds to an accessible boundary point of D at infinity; (3) $w(0)=0$. Conversely, the existence of such a w implies that E is of logarithmic capacity zero. Related results are also given. *M. Heins* (Providence, R. I.).

Ozawa, Mitsuru. On bounded analytic functions and conformal mapping. II. Kōdai Math. Sem. Rep. **1950**, 109–112 (1950).

The author generalizes the algorithm of Schur for studying the coefficients of a bounded analytic function to include the case of multiply-connected domains, but his results are not sharp. Extremal properties of canonical conformal mappings on domains bounded by circular and radial slits are investigated, and special attention is given to the symmetry of triply-connected regions.

P. R. Garabedian (Stanford University, Calif.).

Haegi, Hans R. Extremalprobleme und Ungleichungen konformer Gebietsgrössen. Compositio Math. **8**, 81–111 (1950).

[Also issued as a thesis by the Eidgenössische Technische Hochschule in Zürich, 1950]. Let G be a finite simply-connected domain in the complex plane whose complement Γ is also a simply-connected domain. If $g(z)=\bar{r}z+\text{const.}+O(z^{-1})$ maps $|z|>1$ conformally onto Γ, $\bar{r}$ is called the exterior radius of G. The inner radius r of G is defined as the sup of all numbers r_a such that $f(z)=a+r_az+O(z^2)$, $r_a>0$, maps $|z|<1$ conformally onto G. The author discusses a number of extremal problems involving $\bar{r}$, r, and Euclidean quantities connected with G. The main result of the paper concerns the extremal problem: for given $\bar{r}$ and r, to find the domain G whose boundary has the largest possible diameter. By proper use of the Schiffer variation technique and the Pólya-Szegö symmetrization theorem, the author obtains a complete solution of this problem.

Z. Nehari.

Bergman, S., and Schiffer, M. Kernel functions and conformal mapping. Compositio Math. **8**, 205–249 (1951).

This is a continuation of the work on kernel functions done by the two authors both jointly and individually. If D is a finite domain in the complex z-plane which is bounded by a finite number of closed analytic curves C, the Bergman kernel function $K(z, \zeta)$ $(\zeta\varepsilon D)$ of D is characterized by the reproducing property (*) $\iint_D K(z, \zeta)f(z)dxdy=f(\zeta)$, where $f(z)$ is any function which is regular and single-valued in D and has a finite integral $\iint_D|f(z)|^2dxdy$. The anomaly that all functions of this class are eigenfunctions of the integral equation (*) belonging to the eigenvalue 1 is due to the fact that $K(z, \zeta)$ has a double pole if both z and ζ are identified with the same point of C. Thus $K(z, \zeta)$ is a singular kernel and the Hilbert theory of integral equations does not apply. The authors show that the kernel $K(z, \zeta)-\Gamma(z, \zeta)$, where $\pi^2\Gamma(z, \zeta)=\iint_{D'}(w-z)^{-2}(\bar{w}-\bar{\zeta})^{-2}dudv$ $(w=u+iv)$ and D' is the complement of D, is free from this defect. This kernel is further shown to be Hermitian and positive definite and it thus has a countable number of eigenvalues λ_ν and corresponding eigenfunctions $\varphi_\nu(z)$ for which

$$(^{**})\qquad \varphi_\nu(z)=\lambda_\nu\int\int_D[K(z, w)-\Gamma(z, w)]\varphi_\nu(w)dudv.$$

An integral equation closely allied to (**) is obtained by considering the function $L(z, \zeta)=(z-\zeta)^{-2}+l(z, \zeta)$, introduced earlier by Schiffer, which is connected with $K(z, \zeta)$ by the boundary relation $\overline{K(z, \zeta)dz}=L(z, \zeta)dz$ $(z\varepsilon C)$. It is shown that $l(z, \zeta)$ is regular in the closure of D even if ζ is on C and that the solutions of the integral equation

$$\varphi_\nu(z)=\mu_\nu\int\int_D l(w, z)\varphi_\nu(w)dudv$$

coincide with those of (**), with $\lambda_\nu=\mu_\nu^2$. In the case in which D is simply connected, it is found that both D and its complement have the same set of eigenvalues. Since the kernel $K(z, \zeta)$ is intimately related to most of the canonical conformal mapping functions of D, it is important to devise procedures for the numerical computation of $K(z, \zeta)$. To Bergman's earlier method of computing kernels by orthonormalization of suitable complete sets of functions, the authors now add a successive approximation process based on the integral equation (**) and the fact that the eigenvalues λ_ν satisfy $\lambda_\nu>1$. The essential step of this process is the successive computation of the integrals

$$\Gamma^{(n+1)}(z, \zeta)=\int\int_D\Gamma^{(n)}(z, w)\Gamma(w, \zeta)dudv,\quad \Gamma^{(1)}(z, \zeta)=\Gamma(z, \zeta).$$

The convergence of the method is shown to be geometrical and this may justify the hope expressed by the authors that the process might be useful for the numerical computations of mapping functions. The paper concludes with applications of these concepts to the theory of univalent functions and with the derivation of variation formulas for the eigenfunctions of (**).

Z. Nehari (St. Louis, Mo.).

Yosida, Tokunosuke. On the mapping functions of Riemann surfaces. J. Math. Soc. Japan **2**, 125–128 (1950).

Let W denote a simply-connected Riemann surface whose singularities are logarithmic and lie over a finite number of points of the extended plane, say $x_1, \cdots, x_n$ $(n\geqq 3)$. Let φ denote a function which maps the finite plane (or interior of the unit circle) one-to-one and directly conformally onto W and let m denote a uniformization mapping of the interior of the unit circle onto the extended plane less the points x_k. The author considers the analytic functions f with domain the interior of the unit circle which satisfy $\varphi \circ f=m$. He shows that, if W is parabolic, then $T(r, f)=O[\log(1-r)^{-1}]$, $\neq O(1)$. He asserts that, if W is hyperbolic, then f is a Blaschke product. This, however, need not be the case.

M. Heins (Providence, R. I.).

Radojčić, M. Certains critères concernant le type des surfaces de Riemann à points de ramification algébriques. Acad. Serbe Sci. Publ. Inst. Math. **3**, 25–52, 305–306 (1950).

This paper deals with the type problem in its classical form: conformal equivalence of a given simply-connected Riemann surface with a circle or a punctured plane. The problem is thus far unsolved; some sufficient conditions are known characterizing the strength of the boundary in terms of the length of curves or disc chains approaching the boundary or by the number of branch points or sheets in an exhaustion. The last type of criteria has been restricted to the case of a finite number of projection points of branch points. The author gives the first criterion for surfaces with infinitely many projection points, subject, however, to the following restrictive properties: (1) All branch points are algebraic; (2) the order of the branch points does not exceed a fixed constant p; (3) the spherical distance between any two branch points is at least as large as a fixed constant ϵ. The conventional division of the surface in half-sheets is replaced by division in sheets created by cuts along radial segments connecting the branch points with the infinite point. Let G_0 be one of the sheets and G_ν the νth generation, i.e. the collection of sheets neighboring $G_{\nu-1}$ $(G_\nu\cap G_{\nu-2}=0)$. Denote by $\delta(\nu)$ the number of sheets in G_ν. If

$$\sum_{\nu=1}^{\infty}\frac{\nu}{\delta(0)+\delta(1)+\cdots+\delta(m_\nu)}=\infty,$$

where m_ϵ depends on ν, p, and ϵ, then the surface is parabolic. The theorem is proved as an application of the Ahlfors criterion $\int^\infty(\rho/n(\rho))d\rho=\infty$ [Ann. Soc. Sci. Fennicae Comment. Phys.-Math. 9, no. 6, 1–5 (1936)]. Some variations of the theorem are given. In the second part (pp. 305–306) the author, correcting his earlier (p. 47) erroneous reasoning, shows that the criterion is applicable to the Riemann surfaces of the inverse functions of elliptic functions.

L. Sario (Princeton, N. J.).

Bader, Roger, et Parreau, Michel. Domaines non compacts et classification des surfaces de Riemann. C. R. Acad. Sci. Paris 232, 138–139 (1951).

The authors consider on a Riemann surface S regions which are not relatively compact and have piecewise analytic frontiers. Such a region D is said to belong to the class $\mathfrak{B}$ ($\mathfrak{D}$) provided that there exists in D a harmonic function $u\not\equiv$const. which vanishes on the frontier of D and is bounded (or possesses a finite Dirichlet integral in D). The following results are established: (1) If S can be divided into two regions D' and D'', each of which is not relatively compact, by a set of curves γ and if $D', D''\epsilon\mathfrak{B}$ ($\mathfrak{D}$) then $S\epsilon C_{HB}$ (C_{HD}). The converse also holds. (2) $C_{HM_1}=C_{HB}$ [cf. Parreau, same C. R. 231, 679–681 (1950); these Rev. 12, 259].

M. Heins (Providence, R. I.).

Nevanlinna, Rolf. Beschränktartige Potentiale. Math. Nachr. 4, 489–501 (1951).

This paper is concerned with harmonic functions on a Riemann surface which are representable as the difference of non-negative harmonic functions. Conditions are given which extend known results for the plane case. In addition, the author considers the class E_φ of functions u harmonic on a Riemann surface and such that $\varphi(\hat{u})$ has a finite harmonic majorant where φ has as its domain the set of reals and is convex and increasing. It is shown that such u are representable as the difference of non-negative harmonic functions. An example due to Ahlfors shows that the class of Riemann surfaces admitting non-trivial bounded harmonic functions is a proper part of the class of surfaces with positive boundary. The relation between these two classes and the class of Riemann surfaces for which E_φ is non-trivial remains unsettled. Various remarks are made in connection with this question.

M. Heins (Providence, R. I.).

Kodaira, Kunihiko. Green's forms and meromorphic functions on compact analytic varieties. Canadian J. Math. 3, 108–128 (1951).

The theorem on the existence of an integral of the third kind with assigned singularities on a Riemann surface, when the sum of the residues of the singularities is zero, has been extended to general Kähler manifolds by Weil [Comment. Math. Helv. 20, 110–116 (1947); these Rev. 9, 65] by means of the theory of harmonic integrals. Weil's proof does not give a method of constructing the integral with assigned periods explicitly, and the present paper gives another proof of Weil's result, using the methods of potential theory. The double p-forms $\gamma^p(x,\xi)=\gamma^p(\xi,x)$, which satisfy the equation $\Delta_x\gamma^p(x,\xi)=\sum h_i^p(x)h_i^p(\xi)$, where h_i^p, $i=1,2,\cdots$, is a normalised basis for the harmonic p-forms on a manifold, introduced by de Rham, play a fundamental role in the proof, and the integral of the third kind which has as singular locus a given analytic bounding $(2n-2)$-cycle Γ is expressed in terms of these forms. Necessary and sufficient conditions are then obtained for the existence of a one-valued meromorphic function on the manifold having Γ as its divisor, in a form which includes Abel's theorem on a Riemann surface as a special case. An application of the results is made to the generalised torus.

W. V. D. Hodge (Cambridge, England).

Hawley, Newton S. A theorem on compact complex manifolds. Ann. of Math. (2) 52, 637–641 (1950).

Bochner und Montgomery haben gezeigt [Ann. of Math. (2) 48, 659–669 (1947); diese Rev. 9, 174] dass bei einer kompakten komplexen Mannigfaltigkeit M die Gesamtgruppe der Automorphismen (analytische Homöomorphismen) eine komplexe Liesche Gruppe bilden. Verf. zeigt darüber hinaus, dass eine kompakte komplexe Mannigfaltigkeit höchstens endlich viele Automorphismen besitzt, wenn ihre universelle Überlagerungsmannigfaltigkeit einem Picardschen Gebiet analytisch äquivalent ist. Dabei versteht Verf. unter einem Picardschen Gebiet ein Gebiet des Raumes der komplexen Veränderlichen $z_1,\cdots,z_n$, das mindestens über je zwei Punkten in jeder Koordinatenebene keine inneren Punkte aufweist.

H. Behnke (Münster).

Nikol'skiĭ, S. M. Some inequalities for entire functions of finite degree of several variables and their application. Doklady Akad. Nauk SSSR (N.S.) 76, 785–788 (1951). (Russian)

Let $x=(x_1,\cdots,x_n)$ be a point of Euclidean n-space, $\|g(x)\|_p$ the usual L_p-norm, $G^p(\nu_1,\cdots,\nu_n)$ the class of entire functions $g(z)=g(z_1,z_2,\cdots,z_n)$ satisfying $g(x)\epsilon L_p$, $|g(z)|<\exp\{\sum_{j=1}^n(\nu_j+\epsilon)|z_j|\}$. The subclass $H^p(r,x_1)$ of L_p is defined as follows. If r is an integer, $f\epsilon H^p(r,x_1)$ if $\partial^{r-1}f/\partial x_1^{r-1}=g(x)$ exists for almost all $(x_2,\cdots,x_n)$ and

$$\|g(x_1+h,x_2,\cdots,x_n)+g(x_1-h,x_2,\cdots,x_n)\\ -2g(x_1,x_2,\cdots,x_n)\|\leqq M|h|.$$

If r is not an integer, $r=[r]+\alpha$, $0<\alpha<1$, then $h=\partial^{[r]}f/\partial x_1^{[r]}$ exists for almost all $(x_2,\cdots,x_n)$ and

$$\|h(x_1+h,x_2,\cdots,x_n)-h(x_1,\cdots,x_n)\|_p\leqq M|h|^\alpha.$$

Then $H^p(r_1,\cdots,r_n)=\bigcap_j H^p(r_j,x_j)$. Define $A(f;\nu_1,\cdots,\nu_n)$ as $\inf\|f-g\|_p$, where g runs through all functions of $G^p(\nu_1,\cdots,\nu_n)$. The following results are stated. Theorem 1. $f\epsilon H^p(r_1,\cdots,r_n)$ $(r_j>0)$ if and only if

$$A(f;\nu_1,\cdots,\nu_n)\leqq K\sum_{j=1}^n\nu_j^{-r_j}$$

for all ν_j. Theorem 2. $f\epsilon H^p(r,x_1)$ if and only if there is a sequence of entire functions $\{g_\nu(z_1,\cdots,z_n)\}$ where g_ν is of type ν in z_1 for almost all $(x_2,\cdots,x_n)$ such that

$$\|f-g_\nu\|_p<K\nu^{-r}\qquad(\nu=1,2,\cdots).$$

Theorem 3. If $f\epsilon H^p(r_1,\cdots,r_n)$ and $1\leqq p<s\leqq\infty$ and $k=1-((1/p)-(1/s))\sum_{j=1}^n r_j^{-1}>0$, then $f\epsilon H^s(\rho_1,\cdots,\rho_n)$, $\rho_j=kr_j$. The ρ_j cannot be replaced by larger constants. The proofs are based on the following lemmas. Lemma 1 (S. Bernstein). If $g(z_1,\cdots,z_n)$ is an entire function of exponential type ν for almost all $(x_2,\cdots,x_n)$, then $\|\partial g/\partial x_1\|_p\leqq\nu\|g\|_p$ $(1\leqq p\leqq\infty)$. Lemma 2. If $g\epsilon G^p(\nu_1,\cdots,\nu_n)$, $\alpha_j>0$, $h_j=\alpha_j/\nu_j$, then

$$\|g\|_p\leqq\Big(\prod_{j=1}^n h_j\sup_{(t)}\sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty}|g(k_1h_1-t_1,\cdots,k_nh_n-t_n)|^p\Big)^{1/p}\\ \leqq\prod_{j=1}^n(1+\alpha_j)\|g\|_p,$$

where the summation is over all k_j. Lemma 3. If

$$g\epsilon G^p(\nu_1, \cdots, \nu_n), \quad 1 \leqq p < s \leqq \infty,$$

then $g\epsilon G^s(\nu_1, \cdots, \nu_n)$ and $\|g\|_s \leqq 2^n(\prod_1^n \nu_j)^{(1/p)-(1/s)}\|g\|_p$.

W. H. J. Fuchs (Ithaca, N. Y.).

Lukomskaya, M. A. On a generalization of a class of analytic functions. Doklady Akad. Nauk SSSR (N.S.) **73**, 885–888 (1950). (Russian)

The author extends the main theorems of sigma-monogenic functions [Bers and Gelbart, Trans. Amer. Math. Soc. **56**, 67–93 (1944); these Rev. **6**, 86] to functions $f(z) = u(x, y) + iv(x, y)$, where u and v are connected by the equations (1) $c_j u_x + e_j v_x = a_j u_y + b_j v_y$, $j = 1, 2$, and the coefficients a, b, c, e satisfy the relations (2a) $c_{j,x} = a_{j,y}$, $e_{j,x} = b_{j,y}$ and (2b) $e_2 a_1 = b_2 c_1$, $a_1(e_1 - c_2) = c_1(b_1 - a_2)$, $b_2(e_1 - c_2) = e_2(l_1 - a_2)$. For such functions one can define Σ-integration, Σ-differentiation, formal powers, and prove the analog of Taylor's theorem. The author mentions as an example the equations (3) $\cos\eta \cdot x_\xi - \sin\eta \cdot y_\xi = -y_\eta$, $\sin\eta \cdot x_\xi + \cos\eta \cdot y_\xi = -x_\eta$ which occur in the theory of plasticity. [In (3) the coefficients depend only on one independent variable. The reviewer suspects that in view of (2) equations (1) can always be transformed into such a system.] *L. Bers.*

Theory of Series

Stark, M. On a ratio test of Frink. Colloquium Math. **2**, 46–47 (1949).

In a manner similar to the reviewer's proof that a series of positive terms $\sum a_n$ converges if $\limsup (a_n/a_{n-k})^n < e^{-k}$, the author proves convergence if $\limsup n(a_n/a_{n-k} - 1) < -k$. This is an extension of Raabe's test. It is shown that the two tests are equivalent. The corresponding test for divergence when $n(a_n/a_{n-k} - 1) \geqq -k$ for $n > N$ is shown to be stronger than the reviewer's version $(a_n/a_{n-k})^n \geqq e^{-k}$ for $n > N$. *O. Frink* (State College, Pa.).

***Vernotte, Pierre. Nouvelles recherches sur la sommation pratique des séries divergentes. Aperçus théoriques nouveaux.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 238, viii+278 pp. (1950).

This is a sequel to the author's earlier book [Théorie et pratique des séries divergentes . . ., same Publ., Paris, no. 207 (1947); these Rev. **11**, 97]. There is no improvement in clarity and precision, the many calculations involving such series as $1! - 2! + 3! - 4! + \cdots$ continuing to be accompanied by discussion involving vague ideas and undefined terms.

R. P. Agnew (Ithaca, N. Y.).

Calabi, E., and Dvoretzky, A. Convergence- and sum-factors for series of complex numbers. Trans. Amer. Math. Soc. **70**, 177–194 (1951).

Let Z be a set of complex numbers; Z is called a convergence factor set if for any sequence of complex numbers $\{a_n\}$ with (1) $\lim_{n\to\infty} a_n = 0$ there is a sequence $\{\varphi_n\}$, $\varphi_n \epsilon Z$, such that $\sum \varphi_n a_n$ is convergent, and Z is called a sum factor set, if for any sequence $\{a_n\}$ satisfying (1) and $\sum |a_n| = \infty$ and for any assigned s there is a sequence $\{\varphi_n\}$, $\varphi_n \epsilon Z$, for which $\sum \varphi_n a_n = s$. Theorem 1. A bounded set Z is a convergence factor set if and only if 0 belongs to the closure of the convex hull of Z. Theorem 2. A set Z is a sum factor set if and only if 0 is an interior point of the convex hull of Z. Generalizations to the case that φ_n is chosen from a set Z_n are given. Theorem 2 is also generalized to show that the set of limit points of partial sums of $\sum \varphi_n a_n$ can be any preassigned continuum, if 0 is inside the convex hull of Z.

W. H. J. Fuchs (Ithaca, N. Y.).

Zeller, Karl. Allgemeine Eigenschaften von Limitierungsverfahren. Math. Z. **53**, 463–487 (1951).

The author shows that the use of "spaces (F)" (which are special cases of spaces of type (F) of Banach) and in particular of coordinate spaces (FK) rather than of Banach spaces is the natural way for obtaining several general theorems on methods of summation. Spaces (F) are linear spaces with a topology defined by a sequence of quasinorms $0 \leqq \rho_i(x) < +\infty$ with $\rho_i(ax) = |a|\rho_i(x)$, $\rho_i(x+y) \leqq \rho_i(x) + \rho_i(y)$. Spaces (FK) are spaces (F) whose elements are complex sequences $x = (x_n)$, x_n being a continuous function of x. The main theorem used is the representation of a continuous linear functional $f(x)$ on a space (F), $f(x) = \sum_{i=1}^p f_i(x)$, $|f_i(x)| \leqq M_i \rho_i(x)$. The author defines an (FK) topology in the set $\mathfrak{S}_c A$ of all A-summable sequences ($A, B, \cdots$ denote matrix methods of summation). The following theorems are proved: (i) If C is not weaker than $A = (a_{n\nu})$ and $\chi(A) = \lim_n \sum_\nu a_{n\nu} - \sum_\nu \lim_n a_{n\nu} \neq 0$, $\chi(C) \neq 0$, then there is a method B, consistent with C such that $\mathfrak{S}_c B = \mathfrak{S}_c A$; if $\chi(A) \neq 0$, $\chi(C) = 0$, there is no such B. (ii) Let B be not weaker than A and $\lim_n \sum_\nu a_{n\nu} = \lim_n \sum_\nu b_{n\nu}$, $\lim_n a_{n\nu} = \lim_n b_{n\nu}$, $\nu = 0, 1, \cdots$, if an $x \epsilon \mathfrak{S}_c A$ is not a point of contact of the set of all convergent sequences in $\mathfrak{S}_c A$, then $B - \lim x_n = A - \lim x_n$; if x is a point of contact, there is a B of this type with $B - \lim x_n \neq A - \lim x_n$. (iii) If A sums a divergent bounded sequence x_n with $x_n = o(d_n)$, $x_n - x_{n-1} = o(c_n)$, then there is an unbounded A-summable sequence y_n with $x_n = o(d_n)$, $y_n - y_{n-1} = o(c_n)$. (iv) If the Tauberian theorem: "$x_n - x_{n-1} = o(c_n)$ implies boundedness for any A-summable sequence x_n" holds for A, then even the theorem "$x_n - x_{n-1} = o(c_n)$ implies convergence of any A-summable sequence x_n" holds. (v) A "sum" (or "union") of denumerably many matrix methods, none of which is the strongest, and the method (C, ∞) of Garten and Knopp [Math. Z. **42**, 365–388 (1937)] are not equivalent to any matrix method. In (i)–(iv), A is assumed to sum all convergent sequences. Some of the above theorems are known in a weaker form; see among others, Mazur and Orlicz [C. R. Acad. Sci. Paris **196**, 32–34 (1933)], J. D. Hill [Bull. Amer. Math. Soc. **50**, 227–230 (1944); these Rev. **5**, 236], and Wilansky [Trans. Amer. Math. Soc. **67**, 59–68 (1949); these Rev. **11**, 243].

G. G. Lorentz (Kingston, Ont.).

Karamata, J. Sur le théorème tauberien de N. Wiener. Acad. Serbe Sci. Publ. Inst. Math. **3**, 201–206 (1950).

In this note, dated May 1939, the author outlines a brief proof of Wiener's Tauberian theorem in three steps, assuming the oscillation of the function $s(x)$ to be suitably restricted and that $tK(t) \epsilon L(-\infty, \infty)$. In one variant $K(t)$ is supposed to be positive. The first step consists in proving that $s(x)$ is bounded under the given assumptions, the second involves passing from the given kernel $K(t)$ to the kernel $\lambda[\sin \lambda t/\lambda t]^4$ of H. R. Pitt. For the final step, the passage from the existence of Pitt's limit to that of $s(x)$, he refers to an earlier paper [J. Reine Angew. Math. **178**, 29–33 (1937)].

E. Hille (New Haven, Conn.).

Fourier Series and Generalizations, Integral Transforms

Sz.-Nagy, Béla. Méthodes de sommation des séries de Fourier. II. Časopis Pěst. Mat. Fys. 74 (1949), 210–219 (1950). (French. Czech summary)

[For part I see Acta Sci. Math. Szeged 12, Pars B, 204–210 (1950); these Rev. 11, 656.] A triangular matrix $\Lambda=(\lambda_{nk})$ is called of type $\tilde{F}$, if for every function f of period 2π and with Fourier coefficients a_n, b_n, the expression $\tilde{\sigma}_n(f;x)=\sum_{k=1}^n\lambda_{nk}(a_k \sin kx-b_k \cos kx)$ (the "means" of the series conjugate to the Fourier series of f) converges to

$$\tilde{f}(x)=-\frac{1}{\pi}\int_{\to 0}^{\pi}[f(x+t)-f(x-t)]\tfrac{1}{2}\cot \tfrac{1}{2}tdt$$

at each point x at which the latter integral exists and which is also a Lebesgue point of f. The author shows that for Λ to be of type $\tilde{F}$ it suffices that (1) $\lim_{n\to\infty}\lambda_{nk}=1$ for every fixed k, and that

$$(2)\quad \sum_{k=0}^{\nu-1}(k+1)\left(\log\frac{n}{k+1}\right)|\Delta^2{}_{nk}| + \sum_{k=\nu}^{n-1}(n-k)\left(\log\frac{n}{n-k}\right)|\Delta^2{}_{nk}|=O(1),$$

where $\nu=[\frac{1}{2}n]$, $\Delta^2{}_{nk}=\lambda_{nk}-2\lambda_{n,k+1}+\lambda_{n,k+2}$ $(\lambda_{n,n+1}=0)$.

A. Zygmund (Chicago, Ill.).

Sz.-Nagy, Béla. Méthodes de sommation des séries de Fourier. III. Acta Univ. Szeged. Sect. Sci. Math. 13, 247–251 (1950).

The author establishes necessary conditions in order that the means $\tilde{\sigma}_n(f;x)$ [see the preceding review] be equiconvergent with the $(C, 1)$ means

$$\sum_{k=1}^{n}(1-k/(n+1))(a_k \sin kx-b_k \cos kx),$$

for every continuous f. These conditions are: condition (1) of the preceding review; condition (2) there, with the sign of absolute value on the left omitted; and the condition $|\lambda_{nk}|<C$.

A. Zygmund (Chicago, Ill.).

Salem, R., et Zygmund, A. La loi du logarithme itéré pour les séries trigonométriques lacunaires. Bull. Sci. Math. (2) 74, 209–224 (1950).

Let $S_N(x)=\sum_1^N(a_k \cos n_kx+b_k \sin n_kx)$, $n_{k+1}/n_k\geqq q>1$, $c_k=(a_k^2+b_k^2)^{\frac{1}{2}}$, $B_N=(\frac{1}{2}\sum_1^N c_k^2)^{\frac{1}{2}}$. The authors prove that if $B_N\to\infty$ and $\max_{1\leqq k\leqq N}c_k=O((B_N^2/\text{loglog } B_N)^{\frac{1}{2}})$ then, for almost every x, $\lim\sup_{N\to\infty}S_N(x)/(2B_N^2 \text{ loglog } B_N)^{\frac{1}{2}}\leqq 1$. This result supercedes the corresponding result of Paley and Zygmund which was established only for sufficiently large q (>3).

M. Kac (Ithaca, N. Y.).

Delange, Hubert. Nouveaux théorèmes pour l'intégrale de Laplace. II. C. R. Acad. Sci. Paris 232, 1176–1178 (1951).

Certain very general Tauberian theorems are obtained for the Laplace transform, the statements of which are too long to be reproduced here. This is a continuation of the author's earlier work [same C. R. 232, 589–591 (1951); these Rev. 12, 497].

I. I. Hirschman, Jr. (St. Louis, Mo.).

Widder, D. V. A symbolic form of the classical complex inversion formula for a Laplace transform. Amer. Math. Monthly 58, 179–181 (1951).

Lawden, D. F. The function $\sum_{n=1}^{\infty}n^rz^n$ and associated polynomials. Proc. Cambridge Philos. Soc. 47, 309–314 (1951).

The author states that for the application of a generalized Laplace transform [Stone, J. Appl. Phys. 18, 414–416 (1947); Philos. Mag. (7) 39, 988–991 (1948); Iowa State Coll. J. Sci. 22, 81–83 (1947); 22, 215–225 (1948); these Rev. 8, 517; 10, 416; 9, 289; 10, 37] to many practical problems a knowledge of the transform of n^r, where r is a positive integer, is of assistance. The transform $f(z)$ of $F(n)$ is defined as $f(z)=\sum_{n=0}^{\infty}F(n)z^{-n-1}$. The present paper develops properties of the functions $\psi r(z)=\sum_{n=1}^{\infty}n^rz^n$, which arise in the application of this generalized transform method to the theory of sampling servomechanisms.

H. P. Thielman.

Agarwal, R. P. Sur une généralisation de la transformation de Hankel. Ann. Soc. Sci. Bruxelles. Sér. I. 64, 164–168 (1950).

The function $J_\lambda^\mu(x)=\sum_{r=0}^{\infty}(-x)^r/r!\Gamma(1+\lambda+\mu r)$ may be taken as the kernel of a functional transformation, and it is shown that the reciprocal kernel can also be expressed in terms of the same function.

A. Erdélyi.

Giaccardi, Fernando. Di una formula integrale dei polinomi di Hermite. Boll. Un. Mat. Ital. (3) 5, 270–273 (1950).

Let $f(x)$ be an entire function whose maximum modulus $M(r)$, $|x|\leqq r$, satisfies the inequality $M(r)<e^{\beta r^2}$, where $\beta<\frac{1}{4}$. Then

$$\int_{-\infty}^{\infty}e^{-x^2}H_k(x)f(x)dx=\pi^{\frac{1}{2}}\sum_{s=0}^{\infty}f^{(2s+k)}(0)/s!2^{2s},$$

where $H_k(x)=(-1)^ke^{x^2}(e^{-x^2})^{(k)}$ is the kth Hermite polynomial.

G. Szegö (Stanford University, Calif.).

Levitan, B. M. On a uniqueness theorem. Doklady Akad. Nauk SSSR (N.S.) 76, 485–488 (1951). (Russian)

Let $\sigma(\lambda)$, $-\infty\leqq\lambda\leqq\infty$, be a complex-valued function of bounded variation over $(-\infty, \infty)$ with $\sigma(\lambda+0)=\sigma(\lambda)$. The author proves the following theorems. Theorem 1. Let $\int_{-\infty}^{\infty}\cos \lambda^{\frac{1}{2}}xd\sigma(\lambda)=0$ for all real x, and let there be a fixed $\alpha<2$ such that for sufficiently large $x>0$

$$\int_{-\infty}^{0}\cos \lambda^{\frac{1}{2}}xd\sigma(\lambda)<\exp(x^\alpha).$$

Then $\sigma(\lambda)=\text{const}$. The proof is based on the Phragmén-Lindelöf theorem. Theorem 2. The representation of a continuous function $f(x)$ in the form $f(x)=\int_c^{\infty}\cos \lambda^{\frac{1}{2}}xd\sigma(\lambda)$, $c>-\infty$, is unique. The latter result is used to give a proof of a similar (known) result for the expansion theorem associated with the problem $y''+(\lambda-q(x))y=0$, $(0\leqq x\leqq\infty)$, $\inf q(x)>-\infty$, $y'(0)-hy(0)=0$.

F. V. Atkinson.

Polynomials, Polynomial Approximations

Vicente Gonçalves, J. L'inégalité de W. Specht. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 1, 167–171 (1950).

Let the polynomial $f(z)=z^n+p_1z^{n-1}+\cdots+p_n$ have the zeros γ_j, $j=1, 2, \cdots, n$, and let $P=1+|p_1|^2+\cdots+|p_n|^2$. Specht [Math. Z. 52, 310–321 (1949); these Rev. 11, 431] has proved that, if α is the product of any m of the γ_j $(m=1, 2, \cdots, n)$, then $|\alpha|^2\leqq P$. In the present paper, the

author shows that, if β is the product of the remaining $n-m$ of the γ_j, then $|\alpha|^2+|\beta|^2\leqq P$. That this inequality is the best possible is seen from the example $f(z)=z^n-1$. The proof of this theorem is based upon the factorization $f(z)=d(z)g(z)$ with $d(z)$ having the first mentioned m of the γ_j as its zeros and upon the relations between the P and the coefficients of $d(z)$ and $g(z)$. *M. Marden* (Milwaukee, Wis.).

Specht, Wilhelm. Abschätzung der Wurzeln algebraischer Gleichungen. II. Math. Z. **53**, 357–363 (1950).

In his first paper [Math. Z. **52**, 310–321 (1949); these Rev. **11**, 431], the author showed that if ξ_k denote the zeros of the polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ with $|\xi_1|\geqq|\xi_2|\geqq\cdots\geqq|\xi_n|$, then $|\xi_1\xi_2\cdots\xi_k|\leqq A_1$ where

$$A_1^2=1+|a_1|^2+\cdots+|a_n|^2.$$

The author now shows this theorem to be a special case of one that involves the determinants $A_0=1$ and

$$D_m=A_m^2=\begin{vmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1}\\ \bar\alpha_1 & \alpha_0 & \alpha_1 & \cdots & \alpha_{m-2}\\ \bar\alpha_2 & \bar\alpha_1 & \alpha_0 & \cdots & \alpha_{m-3}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \bar\alpha_{m-1} & \bar\alpha_{m-2} & \bar\alpha_{m-3} & \cdots & \alpha_0\end{vmatrix},\quad m=1,2,\cdots,$$

where $\alpha_p=a_p+\bar a_1a_{p+1}+\bar a_2a_{p+2}+\cdots+\bar a_na_{p+n}$, with $a_0=1$ and $a_r=0$ for $r>n$. The generalized inequality on zeros of $f(z)$, $|\xi_1\xi_2\cdots\xi_k|\leqq(A_m/A_{m-1})\leqq A_m^{1/m}$ is derived by finding the minimum length of a vector in an $(m-1)$-dimensional space associated with the product of $f(z)$ with an arbitrary polynomial of degree $(m-1)$. A further theorem proved is that, if $\eta=|\xi_1\xi_2\cdots\xi_p|$ or $\eta=1$, according as $f(z)$ has in $|z|>1$ exactly p or no zeros, then as $m\to\infty$

$$\lim\,(A_m/A_{m-1})=\lim\,(A_m)^{1/m}=\eta.$$

M. Marden (Milwaukee, Wis.).

Schumacher, Karl Siegfried. Über das asymptotische Verhalten der Wurzeln einer algebraischen Gleichung mit zielstrebigen Koeffizienten. Arch. Math. **2**, 267–272 (1950).

Let S denote the sector: $|z|>R$, $\varphi\leqq\arg z\leqq\Phi$. Let $b_j(w)$ denote functions which are continuous in S and for which $b_j(w)=\beta_j+o(1)$, $j=1,2,\cdots,n$, where β_j are constants such that the equation $z^n+\beta_1z^{n-1}+\cdots+\beta_n=0$ has the distinct roots η_k, $k=1,2,\cdots,n$. Then, as the author shows, the roots $z_k(w)$ of the equation $P(z,w)=z^n+b_1(w)z^{n-1}+\cdots+b_n(w)=0$, when suitably paired with the η_k, satisfy in S the relation $z_k(w)=\eta_k+o(1)$. As he shows further, if each $b_j(w)$ has a continuous derivative $b_j'(w)$ in S with $b_j'(w)=O(f(w))$ and $b_j(w)=\beta_j+O(g(w))$, where $f(w)$ is a real-valued, positive, monotonically decreasing function of $|w|$ and where $g(w)=\int_{|w|}^{\infty}f(u)du$, then $z_k'(w)=O(f(w))$ and

$$z_k(w)=\eta_k+O(g(w)).$$

The proofs are based upon Cauchy's formula for the number of zeros of an analytic function in a circle and upon lemmas on the continuity and asymptotic behavior of solutions of systems of linear equations. *M. Marden.*

Sz.-Nagy, Gyula. Verallgemeinerung der Derivierten in der Geometrie der Polynome. Acta Univ. Szeged. Sect. Sci. Math. **13**, 169–178 (1950).

The author shows that a number of well-known theorems on the zeros of the derivative of a polynomial

$$f(z)=(z-z_1)^{q_1}(z-z_2)^{q_2}\cdots(z-z_m)^{q_m}$$

hold equally well for the zeros of the "generalized derivative", $F(z)=f(z)\sum_1^m p_k(z-z_k)^{-1}$, where the p_k are any positive constants such that $p_1+p_2+\cdots+p_m=q_1+q_2+\cdots+q_m$. [Reviewer's note: In his papers over a period of many years, the reviewer has used the above property concerning $F(z)$ in relation to $f'(z)$; The Geometry of the Zeros of a Polynomial in a Complex Variable, Math. Surveys, no. 3, Amer. Math. Soc., New York, 1949, p. 19; these Rev. **11**, 101.]

M. Marden (Milwaukee, Wis.).

Kac, A. M. On the criterion of aperiodic stability. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 120 (1951). (Russian)

Determinantal criteria are given, characterizing polynomials whose zeros all have negative real parts.

J. G. Wendel (New Haven, Conn.).

Lur'e, O. B. On transfer processes in systems defined by linear differential equations of the 3d and 4th order with constant coefficients. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 113–115 (1951). (Russian)

Some highly specialized remarks about polynomials of third and fourth degree. *J. G. Wendel.*

Radon, Johann. Zur Polynomentwicklung analytischer Funktionen. Math. Nachr. **4**, 156–157 (1951).

Proof of a special case of a theorem of S. Bernstein [J. Math. Pures Appl. (9) **15**, 345–358 (1936)] (which the author seems not to know) on Bernstein polynomials of analytic functions. *G. G. Lorentz* (Kingston, Ont.).

Gel'fond, A. O. On quasi-polynomials deviating least from zero on the segment [0, 1]. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 9–16 (1951). (Russian)

Let M_n be the maximum of the absolute value of that "quasi-polynomial" $\sum_{k=0}^n a_kx^{\lambda_k}$, $a_n=1$, which deviates least from zero on [0, 1], where $0\leqq\lambda_0<\lambda_1<\cdots<\lambda_n$ are given. The author obtains the lower bound

$$M_n>t^{\frac{1}{2}}(2\lambda_n+t)^{\frac{1}{2}}\prod_{k=0}^{n-1}(\lambda_n-\lambda_k)/\prod_{k=0}^{n}(\lambda_n+\lambda_k+t),$$

where t is any positive number, and a more complicated upper bound. *R. P. Boas, Jr.* (Evanston, Ill.).

Pollaczek, Félix. Familles de polynomes orthogonaux avec poids complexe. C. R. Acad. Sci. Paris **232**, 29–31 (1951).

The polynomials $P_n(z)$ defined by the recursion formula

$$p_0(n)P_n(z)-[zp_1(n)+p_2(n)]P_{n-1}(z)+p_3(n)P_{n-2}(z)=0;$$

$P_0=1$, $P_{-1}=0$, are studied where $p_i(t)$ are given polynomials of degree m (except p_1 which is of degree $m-1$) with appropriate highest coefficients. The existence of an analytic function $\chi(z)$ is shown such that the orthogonality relations $\int P_n(z)P_m(z)\chi(z)dz=0$, $m\neq n$, hold. The integration is extended over a closed path surrounding the interval -1, $+1$. This function $\chi(z)$ turns out to be multivalued with -1, and $+1$ as branch points and the integration can be reduced to one along the interval -1, $+1$. The difference of the limits of $\chi(z)$ on the different sides of the interval yields the weight function which is computed. The resulting orthogonal polynomials generalize the Jacobi polynomials and those which the author has studied in recent notes [same C. R. **228**, 1363–1365 (1949); **230**, 36–37, 2254–2256 (1950); these Rev. **10**, 703; **11**, 432; **12**, 24]. *G. Szegö.*

Ceschino, F. Sur une propriété de certains polynômes d'Appell. Ann. Soc. Sci. Bruxelles. Sér. I. 64, 154–155 (1950).

The coefficient of $a^m/m!$ in the expansion of $\exp(ax-p^{-1}a^p)$, p a positive integer, in powers of a is a polynomial $G_m(x)$. It is shown that for large m the G_m can be represented in terms of the so-called trigonometric functions of higher order, and from this fact the author deduces some results of the zeros of $G_m(x)$. *A. Erdélyi* (Pasadena, Calif.).

Chakravarty, Nalini Kanta. On some relations involving Laguerre polynomial $L_n(z)$. Bull. Calcutta Math. Soc. 42, 172–176 (1950).

The relations derived in this paper are either known or are paraphrases of known formulas. *A. Erdélyi.*

Gatteschi, Luigi. Sull'approssimazione asintotica degli zeri dei polinomi sferici ed ultrasferici. Boll. Un. Mat. Ital. (3) 5, 305–313 (1950).

Following a general procedure of Tricomi the author has established recently [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 8, 399–411 (1949); these Rev. 11, 662] a formula for the rth zero θ_r of the ultraspherical polynomial $P_n^{(\lambda)}(\cos\theta)$, $0<\lambda<1$. In this paper the following simpler and sharper formula is derived:

$$\theta_r=\vartheta_r+\frac{\lambda(1-\lambda)}{2n^2}\left(1-\frac{2\lambda}{n}\right)\cot\vartheta_r+\rho(n,\lambda),$$

where

$$\vartheta_r=\frac{2r+\lambda-1}{2(n+\lambda)}\pi;\quad \left[\frac{n-2\lambda+3}{6}\right]+1\leqq r\leqq\left[\frac{n}{2}\right],\quad n\geqq 10;$$

$$|\rho(n,\lambda)|<158\lambda(1-\lambda)(2n)^{-4}.$$

G. Szegö (Stanford University, Calif.).

Colucci, Antonio. Su qualche proprietà dei polinomi di Legendre. Boll. Un. Mat. Ital. (3) 5, 289–292 (1950).

This note contains various remarks concerning certain recent inequalities involving the Legendre polynomials obtained by Turán, Szegö, and Sansone. One of the tools of the author is the fact that the Hessian $(n-1)[f'(x)]^2-f(x)f''(x)$ is positive provided $f(x)$ is a polynomial of degree n with real and distinct roots. It is also shown that for $\lambda>0$ all roots of $P_n^2+\lambda P_{n-1}P_{n+1}$ are real and simple and situated in the interval $-1,+1$. *G. Szegö.*

Special Functions

***Graeser, Ernst. Einführung in die Theorie der elliptischen Funktionen und deren Anwendungen.** Verlag von R. Oldenbourg, München, 1950. 144 pp. 10 DM.

This is a short introductory text intended for students who are acquainted with the elements of the theory of functions of a complex variable. After an exposition of the theory of the $\wp$-function along traditional lines, a number of conformal mappings associated with elliptic functions are discussed in detail. This is followed by a discussion of the functions on a closed Riemann surface of genus 1 and by the derivation of some properties of the elliptic modular function. Finally, there is a short chapter on the numerical computation of elliptic functions and integrals. The book is well written and well organized. What makes one, however, hesitate to recommend the book as an introduction to elliptic functions is the fact that, apart from a brief mention in a 2-page section, the Jacobian functions are completely ignored. In the light of this omission, the author's claim that the book was written "unter besonderer Berücksichtigung der Anwendungsmöglichkeiten" cannot be regarded as entirely justified. The failure to discuss the Jacobian functions also detracts from the value of the book as a preparation for further reading in the literature on elliptic functions and their applications. *Z. Nehari* (St. Louis, Mo.).

Rumyancev, V. V. On reduction of elliptic integrals to canonical form. Akad. Nauk SSSR. Inženernyĭ Sbornik 5, no. 2, 213–218 (1949). (Russian)

The following elliptic integral is considered: $\int[f(\zeta)]^{-\frac{1}{2}}d\zeta$, where $f(\zeta)$ is a polynomial of the fourth degree with real coefficients. It is shown that this integral can be reduced to the canonical form $\int[(\zeta'^2\pm e_1^2)(\zeta'^2\pm e_2^2)]^{-\frac{1}{2}}d\zeta$ by means of the bilinear complex transformation $(\zeta-l)(\zeta'-m)=n^2$ with real l, m, n. A detailed proof, making use of geometrical arguments, is given. The three cases are treated separately.

H. A. Lauwerier (Amsterdam).

Toscano, Letterio. Gli integrali ellittici completi di prima e seconda specie nel calcolo simbolico. Boll. Un. Mat. Ital. (3) 5, 236–238 (1950).

The author proves those particular cases of the formulas in section 13.2 of Watson's book, A Treatise on the Theory of Bessel Functions [Cambridge University Press, Macmillan, New York, 1944; these Rev. 6, 64] in which the hypergeometric series are expressible as complete elliptic integrals, and obtains some known properties of such integrals. *A. Erdélyi* (Pasadena, Calif.).

Koschmieder, Lothar. Funktionales Rechnen mit allgemeinen Ableitungen. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. 1949, 241–244 (1949).

The author proves the formula

$$\Gamma(\beta)\Gamma(\mu-\beta)F(\alpha;\beta;\gamma;x)=\Gamma(\mu)\int_0^1 F(\alpha,\mu;\gamma;sx)s^{\beta-1}(1-s)^{\mu-\beta-1}ds$$

by fractional differentiation, and gives a similar proof of the corresponding formula for Lauricella's hypergeometric series F_A of n variables. *A. Erdélyi* (Pasadena, Calif.).

Bailey, W. N. On the analogue of Dixon's theorem for bilateral basic hypergeometric series. Quart. J. Math., Oxford Ser. (2) 1, 318–320 (1950).

The formula treated is due to F. H. Jackson [same J., Oxford Ser. (1) 12, 167–172 (1941); these Rev. 3, 238], and is equivalent to

$${}_3\Phi_2\left[\begin{matrix}a, b, c;\\ aq/b, aq/c\end{matrix}\quad q^2a^{\frac{1}{2}}/bc\right]=\frac{(q)_{2n}(b)_n(c)_n(bc)_{2n}}{(q)_n(b)_{2n}(c)_{2n}(bc)_n},$$

where $a=q^{-2n}$, n a positive integer. The author rewrites this in the form

$$(*)\quad {}_3\Psi_3\left[\begin{matrix}b, c, d;\\ q/b, q/c, q/d\end{matrix}\quad q^2/bcd\right]=\frac{\prod(q)\prod(q/bc)\prod(q/bd)\prod(q/cd)}{\prod(q/b)\prod(q/c)\prod(q/d)\prod(q/bcd)},$$

where $\prod(a)=\prod_{m=0}^{\infty}(1-aq^m)$, provided that one of b, c, d is of the form q^{-n}. Using Jackson's analogue of Dougall's

theorem, the author proves that (*) holds even if the series does not terminate, and that a similar result holds when the common product of pairs of parameters is any integral power of q. Formulas for series of type ${}_8\Psi_8$ are also given.

N. J. Fine (Philadelphia, Pa.).

von Schelling, Hermann. A second formula for the partial sum of hypergeometric series having unity as the fourth argument. Ann. Math. Statistics **21**, 458–460 (1950).

If $F_n(a, b; c)$ is the sum of the first n terms of the hypergeometric series $F(a, b; c; 1)$ and

$$G_n(a, b; c) = F_n(a, b; c)/F(a, b; c; 1)$$

it is shown that for a and $c-a-b$ positive integers

$$G_n(a, b; c) = G_{c-a-b}(a, b; a+b+n),$$

a formula which leads to simplifications in urn problems of the Pólya-Eggenburger type. *J. Riordan.*

Buchholz, Herbert. Komplexe Integrale für die parabolischen Funktionen mit dem wesentlich singulären Kern exp $(-z/2\cdot\mathfrak{Tg}s)$. Math. Z. **53**, 387–402 (1950).

The "parabolic functions" are Whittaker's functions $M_{k,m}(z)$ and $W_{k,m}(z)$. These functions can be represented in terms of integrals of the form

$$\int e^{-\frac{1}{2}zt}(1-t)^{-\frac{1}{2}+m-k}(1+t)^{-\frac{1}{2}+m+k}dt.$$

In these integrals the author puts $t=\tanh s$, and studies them in the new form. He also derives integral representations (in the variable s) for products of Whittaker functions, and an expansion of the M-function in a series of Bessel functions. *A. Erdélyi* (Pasadena, Calif.).

Meixner, Josef. Reihenentwicklungen von Produkten zweier Sphäroidfunktionen nach Produkten von Zylinder- und Kugelfunktionen. Math. Nachr. **3**, 193–207 (1950).

Spheroidal wave functions are solutions of

$$(1) \qquad \frac{d}{dz}(1-z^2)\frac{dy}{dz}+\left(\lambda+\gamma^2-\frac{\mu^2}{1-z^2}-\gamma^2z^2\right)y=0.$$

Various expansions of such functions are known and in the present paper the author exhibits most of these as particular instances of a key formula. He defines six solutions of (1) by expansions in terms of Legendre functions of the first and second kind, Bessel functions of the first and second kind, and Hankel functions of the first and second kind, respectively. Then he proceeds to construct an infinite series of products of Legendre and Bessel functions which satisfy the wave equation in spheroidal coordinates ξ, η, φ, and proves that this series satisfies (1), with $z=\xi$ or $z=\eta$, provided that the coefficients satisfy a five-term recurrence relation and λ has one of its characteristic values. The sum of the series is identified as a product of a spheroidal function of ξ and of a spheroidal function of η. Particular values of one of these variables lead to expansions of a single spheroidal function (of the other variable). *A. Erdélyi.*

Döhler, O., und Lüders, G. Über einige unendliche Reihen von Besselschen Funktionen. Z. Angew. Math. Mech. **30**, 382 (1950).

The authors prove from the generating function of Bessel coefficients the result $\sum_{n=-\infty}^{\infty}J_n(z)I_{n-m}(z)=z^m/\Gamma(m+1)$ for integer m. The result is a limiting form of Graf's generalization [Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1944, § 11.3; these Rev. **6**, 64] of Neumann's addition theorem. *A. Erdélyi.*

Szász, Otto. On the relative extrema of Bessel functions. Boll. Un. Mat. Ital. (3) **5**, 225–229 (1950).

Let $\Lambda_\alpha(t)={}_0F_1(\alpha+1; -\frac{1}{4}t^2)$ be the entire function involved in the definition of $J_\alpha(t)$, and let $t_{r,\alpha}$ be the rth positive zero of $\Lambda_\alpha(t)$. The relative extrema of $|\Lambda_\alpha(t)|$ are denoted by $\mu_{r,\alpha}$; they occur at $t_{r,\alpha+1}$. The following monotonic properties are proved: (i) $\mu_{r,\alpha}>\mu_{r+1,\alpha}$ if $\alpha>-\frac{1}{2}$; (ii) $\mu_{r,\alpha}<\mu_{r+1,\alpha}$ if $-1<\alpha<-\frac{1}{2}$; and (iii) $\mu_{r,\alpha}>\mu_{r,\alpha+1}$ if $\alpha>-1$. In all three cases r is a positive integer. *A. Erdélyi.*

Gatteschi, Luigi. Valutazione dell'errore nella formula di McMahon per gli zeri della $J_n(x)$ di Bessel nel caso $0\leq n\leq 1$. Rivista Mat. Univ. Parma **1**, 347–362 (1950).

The formula in question gives for the rth zero of the Bessel function $J_n(x)$, $0\leqq n\leqq 1$, the following estimate: $j_{n,r}=x_r-[(4n^2-1)/8x_r]+O(r^{-3})$, $x_r=\frac{1}{2}(2r+n)\pi-\frac{1}{4}\pi$. The author obtains for the modulus of the O-term the following bound: $r(7.4A^2+1.1A)/2^6(2r+n-1)^3(6r-5)$, $A=|4n^2-1|$. To illustrate the accuracy of this estimate, we mention the following inequality which it furnishes:

$$78.14831676<j_{\frac{1}{4},25}<78.14831705.$$

G. Szegö (Stanford University, Calif.).

Barrucand, Pierre. Sur les fonctions de M. S. Colombo. C. R. Acad. Sci. Paris **232**, 1058–1060 (1951).

The author obtains a number of formulas for the function

$$\mu(k, s)=\int_0^\infty [k^x x^{s-1}/\Gamma(1+x)\Gamma(s)]dx$$

and deduces that μ is an entire function of s (which is clear anyhow). Some of his formulas establish relationships with other functions, notably with $\nu(t)=\mu(t, 1)$ and with $A(t)=\int_0^\infty[\sin x/(t+x)]dx$. *A. Erdélyi.*

Rhodes, P. Fermi-Dirac functions of integral order. Proc. Roy. Soc. London. Ser. A. **204**, 396–405 (1950).

After discussing briefly some problems of theoretical physics in which the functions $F_n(\eta)=\int_0^\infty x^n(e^{x-\eta}+1)^{-1}dx$, $n=0, 1, 2, \cdots$, occur, the author turns to the investigation of these functions. He remarks that $F_0(\eta)=\log(1+e^\eta)$, $F_n(0)=n!(1-2^{-n})\zeta(n+1)$, where ζ is Riemann's zeta function, and $F_n(\eta)=F_n(0)+n\int_0^\eta F_{n-1}(t)dt$ so that in principle the functions may be computed recurrently. He proves that $F_n(\eta)+(-)^{n+1}F_n(-\eta)=S_n(\eta)$, where S_n is an explicitly given polynomial of degree $n+1$ whose coefficients contain $\zeta(2r)$, $r=0, 1, \cdots, [\frac{1}{2}n]$. Since there is a convergent expansion for $F_n(\eta)$ when $\eta<0$, the computation for $\eta>0$ becomes possible. The last section of the paper gives computational details, explicit expressions for $S_1(\eta)$ to $S_4(\eta)$, and a 7D table of $F_n(\eta)/n!$ for $n=1(1)4$ and $\eta=-4.0(0.1)0.0$. [Reviewer's remark. These functions have been studied by a number of mathematicians starting with Lerch in 1887 and Jonquière in 1889. A paper by Truesdell [Ann. of Math. (2) **46**, 144–157 (1945); these Rev. **6**, 152] gives a brief summary of the results and references to earlier literature. Several results derived in the paper under review occur in the older literature, but the tables are new.] *A. Erdélyi.*

Scheen, W. L. Factorial series. Math. Centrum, Amsterdam. Rapport **ZW-1950-006**, 30 pp. (1950). (Dutch)

In this paper the author studies factorial series (f.s.) and some of their generalizations from several points of view.

Much of the work uses generalized beta integrals whose properties are investigated in the report. One such integral is $B(\alpha, \beta; \psi(t)) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\psi(t)dt$, and others, B_1 and B_0, involve complex integrals over the loops $0\cdots(1+)$ and $1\cdots(0+)$. Euler's beta function is $B(\alpha, \beta) = B(\alpha, \beta; 1)$.

One type of generalized f.s. is

$$S = \sum a_k \frac{\partial^m}{\partial x^m} B(\alpha - q, \beta + k)$$

(all summations over $k = 0, 1, 2, \cdots$), and another involves B_1. The well-known inverse f.s. correspond to $q = m = \beta - 1 = 0$. A study reveals that the convergence properties of generalized factorial series are similar to those of f.s. The series S is associated with $B(\alpha, \beta; \varphi(t)[\log(1-t)]^m)$ where $\varphi(t) = t^{-q}\sum a_k(1-t)^k$. Interesting light is thrown on the connection between f.s. and asymptotic expansions. Let $f(z) = \int_0^1 t^{z-1}\varphi(t)dt$ and put $t^\alpha = \tau$ $(\alpha > 0)$. It is then shown how to expand, in a f.s., $\alpha f(z) = \sum a_k B(z/\alpha, k+1)$. The factorial series converges for sufficiently large α. As $\alpha \to 0$, the f.s. becomes an asymptotic expansion in descending powers of z, but the convergence is often lost in this process. For numerical computations there may be a most favorable value of α. There are several examples illustrating this point. Perhaps the most interesting example given is that of

$$-\mathrm{Ei}(-z) = \int_z^\infty x^{-1}e^{-x}dx$$

$$= e^{-z}\left[\frac{1}{z} - \frac{1}{z(z+\alpha)} + \frac{2-\alpha}{z(z+\alpha)(z+2\alpha)} - \cdots\right]$$

where the f.s. converges for $\alpha > \log 2$. A numerical investigation shows that $\alpha = 0.5$, although divergent, leads to better numerical approximations than $\alpha = 1$, $\alpha = 2$ (both convergent expansion), or $\alpha = 0$ (Euler's asymptotic series). Generalized beta integrals and generalized f.s. are related and used to investigate certain integral transforms. With $G(w)$, analytic in a circle $|w| < R$, as kernel function one can study $F(z) = \int_0^1 G(zt^\lambda(1-t)^\mu)g(t)\varphi(t)dt$ or the corresponding transform with a loop integral. For instance, $G(w) = (1-w)^{-1}$, $\lambda = 1$, $\mu = 0$ leads to $F(z) = (2\pi i)^{-1}\int_0^{(1+)}(1-zt)^{-1}\psi(t)dt$, and other examples lead to exponential kernels, expansions of Bessel functions, and other interesting expansions.

A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Tolsted, Elmer. Limiting values of subharmonic functions. Proc. Amer. Math. Soc. 1, 636–647 (1950).

L'auteur généralise dans le plan le résultat de Littlewood qu'une fonction sousharmonique u dans le cercle-unité, dont les moyennes de $|u|$ sur les circonférences concentriques sont bornées, admet presque partout une limite radiale finie. La généralisation consiste essentiellement à remplacer cette limite radiale par une limite selon le segment aboutissant à l'extrémité du rayon et faisant un angle constant avec le rayon. Il n'y a pas toujours presque partout une limite dans un angle axé sur le rayon comme l'avait cru Privaloff et on cite un contre-exemple de Zygmund, avec le potentiel de Green de masses ponctuelles. Il est d'ailleurs aisé de remplacer ces masses par une distribution avec densité, mais l'exemple final que veut donner l'auteur de cette variante ne répond pas tout à fait à la question.

M. Brelot.

Leja, Franciszek. Une méthode d'approximation des fonctions réelles d'une variable complexe. Časopis Pěst. Mat. Fys. 74 (1949), 202–206 (1950). (Polish. French summary)

The author considers a bounded real function $f(z)$, subject to $m \leqq f(z) \leqq M$ and termed boundary function, which he supposes, defined on the boundary B of a plane domain D containing the point $z = \infty$, where B has a positive transfinite diameter. This paper is concerned with certain functions, harmonic outside B, which approximate to $f(z)$ on B. Let W denote a (variable) subset of B consisting of $n+1$ distinct points w (or w^*), let $\sum$ be a sum over the values of w in W and let $\prod^*$ be a product over the values of w in W which are distinct from the value w^*. In terms of the Lagrange polynomial $L(z, w^*) = \prod^*(z-w)/(w^*-w)$, let $F(z, \lambda) = \sum|L(z, w)|\exp[n\lambda f(z)]$ for real $\lambda \neq 0$, and let $f_n(z, \lambda)$ denote the infimum, as W varies and n is kept fixed, of the expression $n^{-1}\log F(z, \lambda)$. The author states without proof the following results, the first of which he observes to be implicitly contained in his earlier paper [Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1936, 79–92]: (I) $\lim f_n(z, \lambda) = f(z, \lambda)$ exists for all z, λ, is harmonic in z outside B, and fulfills in B the inequalities $\lambda m \leqq f(z, \lambda) \leqq \lambda f(z)$; (II) if B does not separate the plane and if $f(z)$ is continuous on B, then $f(z) = \lim \lambda^{-1}f(z, \lambda)$ uniformly for z in B as $\lambda \to 0$; (III) the conclusion of (II) holds also if B separates the plane, provided that, given $\epsilon > 0$, there exists a polynomial $P(z)$ such that $|\exp[f(z)] - |P(z)|| < \epsilon$ throughout B. Finally the author observes that if Δ is the complement of $B+D$, we can choose W so that $|L(z, w)| \leqq 1$ in $B+\Delta$ for each w in W, and hence that $m \leqq \lambda^{-1}f(z, \lambda) \leqq M$ in $B+\Delta$, and he deduces: (IV) with the hypotheses of (III), $\lim \lambda^{-1}f(z, \lambda)$ (as $\lambda \to 0$) exists in $B+\Delta$ and constitutes the harmonic interpolation of $f(z)$ in Δ.

L. C. Young.

Komatu, Yûsaku, and Nishimiya, Han. On a theorem of W. Gustin. Kōdai Math. Sem. Rep. 1950, 67–68 (1950).

The theorem in the title concerns a bilinear integral identity satisfied by pairs of functions harmonic in domains of a Euclidean space [Amer. J. Math. 70, 212–220 (1948); these Rev. 9, 352]. It was originally proved without recourse to series in order to afford series-free demonstrations of certain results in harmonic function theory. It is proved here by use of the series expansion of a harmonic function into surface harmonics. The maximum number of linearly independent surface harmonics of fixed order is also given.

W. Gustin (Princeton, N. J.).

Myškis, A. D. On the solution of a boundary problem of potential theory with a generalization of the concept of boundary. Mat. Sbornik N.S. 26(68), 341–344 (1950). (Russian)

L'auteur étend des résultats classiques sur le problème de Dirichlet (selon Perron et Wiener) en prenant une topologie très générale qu'il a étudiée ailleurs [Mat. Sbornik N.S. 25(67), 387–414 (1949); ces Rev. 11, 382]. Il montre sur trois exemples diverses possibilités.

M. Brelot.

Inoue, Masao. Sur la détermination fonctionelle de la solution du problème généralisé de Dirichlet. Mem. Fac. Sci. Kyūsyū Univ. A. 5, 69–74 (1950).

Let C denote the totality of real functions which are continuous on the bounded frontier F of a given domain D in 3-space, and let $[1/\overline{PQ}]^N = \min[1/\overline{PQ}, N]$. With each point $P \epsilon D$ let there be associated a finite functional $A_P(f)$,

$f \epsilon C$, satisfying (i) $A_P(f)+A_P(g)=A_P(f+g)$, $f, g \epsilon C$; (ii) $A_P(f) \geqq 0$, $f \epsilon C$, and $f \geqq 0$; (iii) $A_P([1/\overline{PQ}]^N) \leqq 1/\overline{PQ}$, for each $Q \epsilon F$, and for all N sufficiently large; (iv) $A_P(1/\overline{PQ}) \geqq h(P)$ for each function $h(P)$ which is harmonic in D and for which $\limsup_{P' \to Q} h(P') \leqq 1/\overline{PQ}$, $P' \epsilon D$. Then the author proves that $A_P(f)$ is the solution of the generalized Dirichlet problem for D, with continuous boundary values f. The author then shows how the Wiener solution obtained by the use of "expanding domains," the Lebesque solution obtained by repeated "averages", the Poincaré solution obtained by "balayage", and the Perron solution obtained by taking the "upper envelope" of a class of subharmonic functions, are all included in his general theorem. *M. Reade.*

Emersleben, Otto. Das Selbstpotential der endlichen äquidistanten Punktreihe. Math. Nachr. 3, 373–386 (1950).

The potential energy of a lattice consisting of n particles situated on a straight line at unit distance from each other, each having unit charge and exerting a force which is proportional to the $-(s+1)$th power of the distance, is

$${}_n\Phi(s)=\sum_{k=1}^{n-1}(n-k)k^{-s}=n\zeta_n(s)-\zeta_n(s-1),$$

where $\zeta_n(s)$ is the nth partial sum of $\sum m^{-s}$. If there are $2n$ points charged alternately with $+1$ and -1, the potential energy is

$${}_{2n}\bar{\Phi}(s)=\sum_{m=1}^{2n}(-)^m(2n-m)m^{-s}=-2n\bar{\zeta}_{2n}(s)+\bar{\zeta}_{2n}(s-1)$$

where $\bar{\zeta}_n(s)$ is the nth partial sum of $\sum(-)^m m^{-s}$. For the Newtonian potential $s=1$. The potential energy of other arrangements can be expressed in terms of Φ and $\bar{\Phi}$. The author obtains asymptotic expansions of ${}_n\Phi(s)$ and ${}_n\bar{\Phi}(s)$ for fixed $s \geqq 0$ as $n \to \infty$, essentially by applying the Euler-Maclaurin summation formula to ζ_n. It turns out that ${}_n\Phi(s)$ is $O(n^2)$ for $s=0$, the order decreases, and is $O(n)$ when $s>1$. On the other hand, ${}_n\bar{\Phi}(s)=O(n)$ for $s \geqq 0$. *A. Erdélyi.*

Differential Equations

Tanturri, Giuseppe. Alcune particolarità proiettive di sistemi ∞^5 di curve nello spazio. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 145–172 (1950).

Considering the five-parameter family F of curves, in three-dimensional space, defined by a system of differential equations of the form $y'''=A(x, y, z, y', z', y'')$, $z''=B(x, y, z, y', z', y'')$, the author studies relations between special geometrical properties of the family of curves and special analytical forms of the functions A, B. Use is made of certain projective concepts and constructions introduced by Bompiani, and the concept of "properties holding in certain approximations" introduced by Terracini. In consequence of this, the results differ considerably from those obtained previously in similar studies by such authors as Comenetz [Amer. J. Math. 58, 225–235 (1936)], DeCicco [Trans. Amer. Math. Soc. 57, 270–286 (1945); these Rev. 6, 186], Kasner [ibid. 8, 135–158 (1907)], and the reviewer [ibid. 60, 149–166 (1946); these Rev. 7, 528].

The results contained in the paper are too numerous to be summarized adequately in a review. Two rather typical results can be described as follows. Consider the one-parameter subfamily of curves which pass through both of two neighboring points M: (x_0, y_0, z_0) and N: (x_1, y_1, z_1). If, for every choice of M and N (the distance between the points being sufficiently small), the tangents to the curves of the subfamily at M lie in a plane, the family F is said to possess property P in the finite sense. The condition for this is the identical vanishing of a certain expression Φ. If Φ is developed in powers of x_1-x_0, the expansion ordinarily begins with a term in $(x_1-x_0)^8$. If the coefficient of this term vanishes, the family F is said to possess property P in at least the first approximation. A necessary and sufficient condition for this is that the function B in the defining differential equations be of the form $B=C(x, y, z, y', z')+D(x, y, z, y', z')y''$. If, in addition, the coefficient of $(x_1-x_0)^9$ in the expansion of Φ vanishes, F is said to possess property P in at least the second approximation. A necessary and sufficient condition for this is that the function D satisfy the relation $\partial D/\partial y'+D\partial D/\partial z'=0$. It should be remarked that the previous studies have related to cases in which $C\equiv 0$, and that the more general discussion in this paper shows that such cases are exceptional in many respects. *L. A. MacColl* (New York, N. Y.).

Richard, Ubaldo. Sulle successioni di valori stazionari delle soluzioni di equazioni differenziali lineari del 2° ordine. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 309–324 (1950).

This investigation is based on a result of Sonin concerning the differential equation $(py')'+qy=0$ which was generalized by Pólya [see Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, p. 161; these Rev. 1, 14]. Assuming that p and q are given positive functions, we denote by x_k, ξ_k the points at which y and y' are stationary, respectively. Under the assumption that pq is monotonic, the monotonic property of the following sequences is established: $|y(x_k)|$, $|p(\xi_k)y'(\xi_k)|$, $(p(x_k)q(x_k))^{\frac{1}{2}}|y(x_k)|$, $|p(\xi_k)y'(\xi_k)|/(p(\xi_k)q(\xi_k))^{\frac{1}{2}}$. (It is also possible to distinguish between the cases of increase and decrease.) Moreover this result is generalized and illustrated by various classical examples. *G. Szegö.*

Hronec, Jur. Les conditions nécessaires et suffisantes pour qu'un système différentiel n'ait pas de points singuliers essentiels. Časopis Pěst. Mat. Fys. 74 (1949), 187–196 (1950). (Czech. French summary)

The author investigates the system $dy_n/dx=\sum_m y_m a_{mn}$, $m, n=1, 2$, and finds conditions for this system to have only regular singularities. He also investigates the series solutions. *A. Erdélyi* (Pasadena, Calif.).

Horošilov, V. V. On solutions of systems of linear differential equations with an irregular singular point. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 37–54 (1951). (Russian)

Proofs are supplied for results previously announced [Doklady Akad. Nauk SSSR (N.S.) 72, 241–242 (1950); these Rev. 11, 664]. *J. G. Wendel.*

Rabinovitch, Norman Louis. Sur les courbes définies par les équations différentielles. C. R. Acad. Sci. Paris 232, 671–673 (1951).

Let $f(x, y)$, $g(x, y)$ be functions which possess continuous partial derivatives of all orders at the point $(0, 0)$, but which are not holomorphic in the neighborhood of that point. Let the functions and their first partial derivatives vanish at $(0, 0)$. This note concerns the nature of $(0, 0)$ as a singular

point of the family of trajectories defined by the system of differential equations $dx/dt=y+f(x,y)$, $dy/dt=-x+g(x,y)$. The author gives two examples which bring out the not very surprising fact that an application of Poincaré's method, based upon a consideration of the formal Maclaurin series associated with $f(x,y)$, $g(x,y)$, without regard to the fact that the series do not represent the functions, may indicate that the singular point is a center even when it is really a focus or a singular point of a higher type.

L. A. MacColl (New York, N. Y.).

Levi, Eugenio. Sul comportamento asintotico delle soluzioni dei sistemi di equazioni differenziali lineari omogenee. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 465–470 (1950).

A result of Faedo [Ann. Mat. Pura Appl. (4) **26**, 207–215 (1947); these Rev. **10**, 120] is improved by the author. In the notation of the review [loc. cit.] γ is replaced by μ, where μ is the maximum of the exponents of the elementary divisors of the characteristic matrix, so that the relevant criterion becomes $\int^\infty x^{\mu-1}|\phi_{ks}|dx<\infty$. [See also Faedo, same Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 564–570, 757–764 (1947); **3**, 37–43 (1947); these Rev. **9**, 285.] *N. Levinson.*

Yakubovič, V. A. On the asymptotic behavior of the solutions of a system of differential equations. Mat. Sbornik N.S. **28(70)**, 217–240 (1951). (Russian)

Vector differential equations of the type $dx/dt=Ax+f(t;x)$ are considered, with A a constant matrix, and f subjected to one of the conditions (a) $\|f(t;x)\|\leqq g(t)\|x\|$, (b) $\|f(t;x+h)-f(t;x)\|\leqq g(t)\|h\|$, where g is a continuous function. Inequalities of simple type for solutions are obtained; these are used to explore the relationship between solutions of the given equation and those of the linear equation obtained by setting $f=0$. Mild generalizations to the case where A is a function of t are given.

J. G. Wendell (New Haven, Conn.).

Erugin, N. P. On the continuation of solutions of differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 55–58 (1951). (Russian)

Standard theorems on the continuation of integral curves are reproved. *J. G. Wendel* (New Haven, Conn.).

Newman, M. H. A. On the ultimate boundedness of the solutions of certain differential equations. Compositio Math. **8**, 142–156 (1950).

Proofs are given, under less restrictive restrictions on g and p, of results of Cartwright and Littlewood [Ann. of Math. (2) **48**, 472–494 (1947); these Rev. **9**, 35] on the equation (1) $\ddot{x}+k\dot{x}f(x)+g(x)=kp(t)$. The trajectories of (1) are compared with those of (2) $\ddot{x}+h\dot{x}+g(x)=0$, $h>0$, and it is shown that when $kf(x)\geqq 2h$ for large x a trajectory of (1) that starts near a trajectory of (2) stays near it except possibly near the origin. *N. Levinson.*

Colombo, Guiseppe. Sull'equazione differenziale non lineare del terzo ordine di un circuito oscillante triodico. Rend. Sem. Mat. Univ. Padova **19**, 114–140 (1950).

The author considers the equation

$$\dddot{x}+(af(x)+b)\ddot{x}+af''(x)\dot{x}^2+a(bf(x)-\alpha)\dot{x}+\beta f(x)=0,$$

treated heuristically by the reviewer [Friedrichs, LeCorbeiller, Levinson, and Stoker, Non-Linear Mechanics, Providence, R. I., 1943]. Subject to the conditions $f(0)=0$, $f'(x)>0$, $xf''(x)\geqq 0$, $x^{-1}f(x)\to\infty$ as $x\to\infty$, $ab^2-4\beta\neq 0$, and $a(af'(0)+b)(bf'(0)-\alpha)-\beta f'(0)<0$, the author gives a rigorous proof of the existence of a periodic solution. The methods are mainly geometric or, more precisely, topological.

N. Levinson (Cambridge, Mass.).

Colombo, Giuseppe. Sulle oscillazioni non-lineari in due gradi di libertà. Rend. Sem. Mat. Univ. Padova **19**, 413–441 (1950).

The author considers first $\dot{x}=v$, $\dot{v}=k(1-x^2)v-\bar{\omega}^2x+e(t)$, $k>0$, $\bar{\omega}^2>0$, and $|e(t)|\leqq\rho<\bar{\omega}^2$ and proves the existence of a closed curve Γ in the (x,v)-plane such that all solutions $(x(t),v(t))$ of the system cross Γ only from the exterior to the interior for increasing t. [The method has been used by the reviewer, J. Math. Physics **22**, 41–48 (1943); these Rev. **5**, 66.] The author then shows how his results can be used to obtain the existence of periodic solutions for certain related systems with two degrees of freedom. *N. Levinson.*

Minorsky, Nicolas. Sur une équation différentielle de la physique. C. R. Acad. Sci. Paris **232**, 1060–1062 (1951).

The existence of stable periodic solutions of the differential equation $\ddot{x}+b\dot{x}+(a-\epsilon x^2)x\cos 2t+ex^3=0$ is discussed by means of the method explained in a previous note by the author [same C. R. **231**, 1417–1419 (1950); these Rev. **12**, 413]. The coefficients b, a, ϵ, and e are small parameters of the same order of magnitude. The case $\epsilon=0$ was studied previously [loc. cit.]. Here a criterion for the general case is given, and some more explicit information is obtained in fourteen special cases. *W. Wasow* (Los Angeles, Calif.).

Minorsky, N. Parametric excitation. J. Appl. Phys. **22**, 49–54 (1951).

The title refers to the excitation of oscillations in a physical system by forcing one of the parameters of the system to vary periodically with respect to time. The author discusses this phenomenon, using a method which can be adapted readily to various particular cases. In the case in which the differential equation of motion is the Mathieu equation $\ddot{x}+(1+\gamma\cos 2t)x=0$, the author introduces the functions $P(t)=\log(x^2+\dot{x}^2)$, $\theta(t)=\arctan(\dot{x}/x)$, and he computes these functions, for $0\leqq t\leqq 2\pi$, by a perturbation method, assuming that γ is small. It is found that there are two values of $\theta(0)$ such that $\theta(2\pi)-\theta(0)=0$, and that, in a certain sense, one of these values is stable and the other unstable. If $\theta(0)$ has the stable value, $P(2\pi)-P(0)$ is approximately $\pi\gamma$. Thus, to the approximation employed, there is an oscillation in which θ is periodic with the period 2π and P increases by an additive constant in the course of a period. A similar discussion is applied to the nonlinear equation $\ddot{x}+p\dot{x}+(1+\gamma\cos 2t)x+\epsilon x^3=0$, where p, γ, and ϵ are small constants of the same order of magnitude; somewhat more complicated results of the same kind are obtained. Some of the results in this part are slightly obscured by the fact that a fundamental system of difference equations has been replaced by a formally similar system of differential equations. This replacement seems to be unnecessary. The author presents experimental results which illustrate some of the conclusions. The paper closes with a discussion of the case in which the differential equation of motion is $\ddot{x}+f(t)x=0$, where the periodic function $f(t)$ is equal to one positive constant in one part of the period, and to another positive constant in the remaining part.

L. A. MacColl (New York, N. Y.).

Malkin, I. On the stability of motion in the sense of Lyapunov. Amer. Math. Soc. Translation no. 41, 68 pp. (1951).

Translated from Rec. Math. [Mat. Sbornik] N.S. 3(45), 47–100 (1938).

Malkin, I. Some basic theorems of the theory of stability of motion in critical cases. Amer. Math. Soc. Translation no. 38, 50 pp. (1951).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 411–448 (1942); these Rev. 4, 225.

Kovalenko, K. R., and Krein, M. G. On some investigations of A. M. Lyapunov on differential equations with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) 75, 495–498 (1950). (Russian)

The authors state without proof a number of results pertaining to the solutions of equations of the type $u''-q(t)u'+ap(t)u=0$, where the coefficients are periodic. These results are related to those of Liapounoff, Krein, Putnam, and Borg [cf. Amer. J. Math. 71, 67–70 (1949); these Rev. 10, 456]. *R. Bellman.*

★Lyapunov, A. M. Obščaya zadača ob ustoĭčivosti dviženiya. [General Problem of the Stability of Motion]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 471 pp. (1 plate).

In addition to the paper of the title [Kharkow, 1892 = Ann. Fac. Sci. Univ. Toulouse (2) 9, 203–474 (1907); cf. these Rev. 9, 34], this volume contains the following papers: Comm. Soc. Math. Kharkow (2) 3, 265–272 (1893); Mat. Sbornik (1) 17, 253–333 (1893); J. Math. Pures Appl. (5) 3, 81–94 (1897).

Nikol'skiĭ, G. N. On a problem of indirect regulation. Akad. Nauk SSSR. Inženernyĭ Sbornik 4, no. 2, 113–132 (1948). (Russian)

This paper is concerned with stability conditions for a closed-cycle control mechanism containing a servomotor of nonlinear characteristic. A typical problem as considered is described by the equations (1) $P(D)y=Q(D)x$, (2) $Dx=\varphi(x+y)$, where x measures the motion of the servomotor, $x+y$ the activating impulse; P, Q are polynomials with constant coefficients in $D=d/dt$, and φ is the nonlinear characteristic of the servomotor, for which $\varphi(0)=0$, $x\varphi(x)\leqq 0$. It is assumed that the servomotor when activated by a harmonic impulse carries out, after a transient, a periodic motion $x=x(t,\tau)$ of period τ and of amplitude $x(\tau,\tau)$. If $y(t,\tau)$ is the periodic solution of (1) for $x=x(t,\tau)$ then the stability condition, derived by the author from physical considerations, is $x(\tau,\tau)+y(\tau,\tau)>0$ for all $\tau>0$. *M. Golomb* (Lafayette, Ind.).

Čekmarev, A. I. Nonlinear oscillations of antivibrators. Akad. Nauk SSSR. Inženernyĭ Sbornik 5, no. 1, 140–157 (1948). (Russian)

The author treats the problem of torsional damping by means of pendulums. It is shown that the problem is expressible in terms of two nonlinear differential equations of the second order, one of which has an external periodic excitation. In these equations the coefficients of the second derivatives are trigonometric functions of dependent variables. Ritz's variational method is used to obtain the expression for the forced oscillation. This reduces the problem to a transcendental equation involving Bessel's functions J_0 and J_1 as well as parameters from which the equivalent moment of inertia of the antivibrator is computed. It is shown that this quantity is nearly constant for small oscillations and is a function of the amplitude for larger ones. In the latter case the method of continuous fractions, attributed to V. P. Terskih, is applied. The method itself is not outlined in the paper. *N. Minorsky.*

Mitrinovitch, D. S. Sur un procédé fournissant des équations différentielles linéaires intégrables d'un type assigné d'avance. Acad. Serbe Sci. Publ. Inst. Math. 3, 227–234 (1950).

The fact that eliminating z from the first order system of differential equations $fy'+gy=z$, $z'+hz=0$, yields the equation $fy''+(f'+g+fh)y'+(g'+gh)y=0$, provides the author with a method for integrating certain special second order differential equations by quadratures. In particular, he outlines the method for certain special cases of the equation $y''+(ae^{sx}+b)y'+(Ae^{2sx}+Be^{sx}+C)y=0$. This equation can be solved in general in terms of Whittaker functions by means of the transformation $y=e^{at/(2s)}t^{-\frac{1}{2}(1+b/s)}v$, $x=(\ln t)/s$, $t=sz/\sqrt{(a^2-4A)}$. *E. Pinney* (Berkeley, Calif.).

Mitrinovitch, Dragoslav S. Sur un procédé d'intégration d'une équation de Monge. C. R. Acad. Sci. Paris 232, 1334–1336 (1951).

The equation treated is $f''/f-\lambda g''/g=\mu(x)$, where λ, μ are known and f, g unknown functions of the independent variable x. The substitution $F=(\log f)'$, $G=(\log g)'$ and elimination of F by $F=\theta+\lambda G$ give a quadratic in G whose solution yields algebraic formulas for F, G in terms of the given λ, λ', μ, arbitrary θ, and θ'. For $\lambda=1-n^2$, $\lambda'=\mu=0$ the equation arose in a problem of elasticity. Generalizations are indicated. *J. M. Thomas* (Durham, N. C.).

Mitrinovitch, D. S. Sur l'équation différentielle d'un problème de Kuhelj. Bull. Soc. Math. Phys. Macédoine 2, 31–34 (1951).

The author notes that the solution of the equation $y''(x)-[B'(x)/B(x)]y'(x)-B(x)y(x)=0$ is

$$y(x)=\exp\left(\int[B(x)/z(x)]dx\right),$$

where $z(x)$ is a solution of the equation $z'(x)+z^2(x)=B(x)$. *E. Pinney* (Berkeley, Calif.).

Försterling, Karl, and Wüster, Hans-Otto. Über die Reflektion in einem inhomogenen Medium. Ann. Physik (6) 8, 129–133 (1950).

In an earlier paper [same Ann. (5) 11, 1–39 (1931)] the differential equation $d^2u/dz^2+\epsilon(z)u=0$ was discussed when $\epsilon(z)=\epsilon_1 z^k$, ϵ_1 constant. The present paper gives a simplified discussion which utilizes certain properties of Bessel functions. *A. Erdélyi* (Pasadena, Calif.).

Viguier, Gabriel. Notions métriques liées à l'équation de Riccati et à l'étude du problème de J. Boussinesq. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 932–936 (1950).

This problem has been similarly treated by the author [C. R. Acad. Sci. Paris 230, 1343–1344 (1950); these Rev. 11, 624]. *Y. H. Kuo* (Ithaca, N. Y.).

Mihailović, Miodrag V. Sur l'intégrale de l'équation différentielle de Thomas-Fermi autour du point $x=0$, $y=1$. Acad. Serbe Sci. Publ. Inst. Math. 3, 259–270 (1950).

By a procedure similar to successive approximations, the author determines asymptotic developments near $x=0$ of

solutions of $y''=x^a y^b$, where $a>-2$, $b>0$, and $y(0)=1$. These developments involve sums of terms of the type $c_{kj}x^{k+ja}$ and a remainder term of the type $O(x^{n(a+2)})$, where k, j, n are integers. The coefficients $c_{kj}=c_{kj}(a, b, C)$ are given explicitly in terms of a, b, and an integration constant C. *P. Hartman* (Paris).

Campbell, R. Sur quelques équations de la physique mathématique. Bull. Sci. Math. (2) **74**, 145–153 (1950).

The author deals with a differential equation of the second order, having a periodic coefficient. He shows that this equation is the prototype of Whittaker's equation, of Lamé's equation, and of the associated Mathieu's equation. He then considers these 3 special cases separately and deals with their solutions by means of infinite sets of linear equations for their coefficients. *M. J. O. Strutt* (Zurich).

Humbert, Pierre, et Delerue, Paul. Sur l'équation différentielle de la fonction de Bessel du troisième ordre, d'indices nuls. Ann. Soc. Sci. Bruxelles. Sér I. **64**, 160–163 (1950).

The authors compute a fundamental system of solutions of the differential equation $y'''+3x^{-1}y''+x^{-2}y'+y=0$ for the neighbourhood of the origin. *A. Erdélyi.*

Krein, M. G. Solution of the inverse Sturm-Liouville problem. Doklady Akad. Nauk SSSR (N.S.) **76**, 21–24 (1951). (Russian)

Denote by $S_q(\alpha, \beta)$ the spectrum of $y''+y(\lambda-q(x))=0$ with boundary conditions

$$\cos \alpha y(0)+\sin \alpha y'(0)=0, \qquad \cos \beta y(l)+\sin \beta y'(l)=0.$$

The author studies "problem A": Given two spectra $S=\{\lambda_n\}$, $S'=\{\lambda_n'\}$, to ascertain the existence of and to determine α, α', β, and $q(x)$ such that $S_q(\alpha, \beta)=S$, $S_q(\alpha', \beta)=S'$. Here the spectra S, S' are to interlace and to satisfy the asymptotic law λ_n, $\lambda_n'\sim\pi^2n^2/l^2$. Theorems of uniqueness for similar problems have been given by G. Borg [Acta Math. **78**, 1–96 (1946); these Rev. **7**, 382], N. Levinson [Mat. Tidsskr. B. **1949**, 25–30 (1949); these Rev. **11**, 248], and L. A. Čudov [Mat. Sbornik N.S. **25(67)**, 451–456 (1949); these Rev. **11**, 248]. The solution rests upon the following "idea." "Just as to every Jacobian matrix J corresponds a certain power-moment problem, for each solution of which (i.e. the distribution-function) the matrix J is itself completely determined, so to a second-order differential operator L with boundary condition on one end corresponds a 'generalised' moment problem, for each distribution-function of which is completely determined, together with the boundary condition, the operator itself . . .". There results a procedure whereby problem A may be solved, certain necessary conditions being cited. Finally there are some remarks on corresponding problems for $y''+\lambda\rho(x)y=0$. *F. V. Atkinson* (Ibadan).

Friedman, Morris D. Determination of eigenvalues using a generalized Laplace transform. J. Appl. Phys. **21**, 1333–1337 (1950).

An Stelle der sogenannten Sommerfeldschen Polygonmethode bei Differentialgleichungen vom Typ

$$[P(x)U'(x)]'-[Q(x)-\lambda R(x)]U(x)=0$$

wird hier eine Methode vorgeschlagen, bei der die ganze Differentialgleichung einer Laplacetransformation unterzogen wird. Bei den hier in Betracht kommenden Eigenwertproblemen, die aus speziellen Fällen der hypergeometrischen und konfluenten hypergeometrischen Reihe entstehen, ergibt sich für die Laplacetransformierte der Funktion $U(x)R(x)=\bar{U}(x)$ eine Differentialgleichung zweiter Ordnung. Fordert man analytisches Verhalten in den singulären Punkten, so erhält man eine Möglichkeit, die Eigenwerte zu bestimmen. Bei diesem Verfahren ergeben sich bei gewissen Beispielen gelegentlich geschlossene Darstellungen für die gesuchten Eigenfunktionen. *P. Funk* (Wien).

Kato, Tosio. Variational methods in collision problems. Physical Rev. (2) **80**, 475 (1950).

The author studies the connection between the variational methods of the reviewer [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 21 (1944); Ark. Mat. Astr. Fys. **35A**, no. 25 (1948); these Rev. **6**, 11; **10**, 120], Schwinger [~~same Rev.~~ Phys. Rev. (2) **72**, 742 (1947); **78**, 135–139 (1950); cf also Blatt and Jackson, ibid. **76**, 18–37 (1949), pp. 20–22], Kohn [ibid. **74**, 1763–1772 (1948)], and Huang [ibid. **76**, 1878–1879 (1949); these Rev. **11**, 464].

Consider the equation (1) $L[u]=d^2u/dr^2+k^2u+W(r)u=0$ with (2) $u(0)=0$, $u(r)\rightarrow\cos kr+\lambda \sin kr$, $r\rightarrow\infty$. Then the functional (3) $F=F(u)=k\lambda-\int_0^\infty uL(u)dr$ is stationary for the exact solution u (with $\lambda=\cot \eta$, η asymptotic phase). This is the basic variational principle [the author appears to have overlooked that it was already announced in the reviewer's paper cited first above]. The author points out that the Schwinger variational method is equivalent to considering, instead of F, the functional (4) $F_S=F+\int_0^\infty[L(u)]^2W^{-1}dr$ and requiring F_S to be stationary. This leads to the same results as (3), since the last term in (4) and its variation vanish for the correct solution. An obvious conclusion [not drawn by the author] is that the last integrand in (4) could be generalized to any polynomial or analytic function $F(L(u))$ with $F(0)=F'(0)=0$, and that the original formulation of the variational principle is the simplest one in the sense that F contains $L(u)$ only linearly. The author emphasizes as an interesting consequence "that the Schwinger method always gives a larger (smaller) value of $k \cot \eta$ than does the reviewer's if $W(r)\geqq0$ ($W(r)\leqq0$) everywhere." A continuation of the paper is announced. *L. Hulthén.*

Schouten, J. A. Regular systems of equations and supernumerary coordinates. Math. Centrum Amsterdam. Scriptum no. **6**, ii+83 pp. (1951).

This pamphlet contains material given in lectures at Amsterdam in 1947. Much of the contents has already appeared in the treatise by the author and van der Kulk [Pfaff's Problem and its Generalizations, Oxford, 1949; these Rev. **11**, 179]. The particular aspects of the subject treated here center around the implicit function theorem and its application to the parametric representation of manifolds defined by equating to zero a finite number of holomorphic functions. *J. M. Thomas* (Durham, N. C.).

Aczél, J. Über Niveaukurven und Flächen von Lösungsfunktionen partieller Differentialgleichungen. Acta Math. Acad. Sci. Hungar. **1**, 125–132 (1950). (German. Russian summary)

For each of the partial differential equations

$$a_1u_x+a_2u_y+a_3=0, \quad a_1u_x+a_2u_y+a_3u_{xx}+a_4u_{xy}+a_5u_{yy}=0,$$

where the a's are given functions of x, y, the paper gives a necessary and sufficient condition that a given family of curves $F(x, y)=\text{const.}$ be contours and determines the corre-

sponding $u(x, y)$ when the condition is met. The analogy to problems solved in potential theory and nomography is pointed out. *J. M. Thomas* (Durham, N. C.).

Butlewski, Zygmunt. Sur les intégrales oscillantes d'une équation différentielle aux dérivées partielles du second ordre. Ann. Soc. Polon. Math. 23, 43–68 (1950).

The oscillation properties of a particular solution of the equation $A(x)r+B(x)p+C(x)z=A_1(y)t+B_1(y)q+C_1(y)z$, in Monge's notation, are studied at length. For

$$B=C=B_1=C_1=0$$

the equation becomes that of a vibrating string with $1/A$ the density per unit length, $1/A_1$ the tension, and y the time. *J. M. Thomas* (Durham, N. C.).

Piskunov, N. S. Solution of a boundary problem for a nonlinear parabolic equation of motion of liquids and gases in a porous medium. Doklady Akad. Nauk SSSR (N.S.) 76, 505–508 (1951). (Russian)

The nonlinear equation

$$2p\frac{\partial^2 p}{\partial r^2}+2\left(\frac{\partial p}{\partial r}\right)^2+2r^{-1}p\frac{\partial p}{\partial r}-a^2\frac{\partial p}{\partial t}=0 \qquad (r>r_0)$$

with conditions $p(r, 0)=P_0$, $\int_0^T p\partial p/\partial r]_{r=r_0}dt=-Q$ describes a problem of flow in a porous medium. The substitution $\xi=r/\sqrt{t}$ enables the author to write the equation in the form $p''/p'+p'/p+1/\xi+a^2\xi/4p=0$ which can be integrated:

$$pp'=c\xi^{-1}\exp\left[-\frac{a^2}{4}\int_{\xi_0}^{\xi}p^{-1}\xi d\xi\right].$$

The new conditions $\lim_{\xi\to\infty}p(\xi)=P_0$,

$$[pp']_{\xi=\xi_0=r_0/\sqrt{T}}=-Q^*\sqrt{T}$$

lead finally to

$$\tfrac{1}{2}p^2=Q^*r_0\int_{\xi}^{\infty}\xi^{-1}\exp\left[-\frac{a^2}{4}\int_{\xi_0}^{\xi}p^{-1}\xi d\xi\right]d\xi+\tfrac{1}{2}P_0^2.$$

An existence proof follows, in the course of which a method of successive approximations for finding p is evolved, and the constant Q^* is evaluated, $Q^*T=Q$. *R. E. Gaskell.*

Krzyżański, M. Sur l'équation aux dérivées partielles de la diffusion. Ann. Soc. Polon. Math. 23, 95–111 (1950).

Denote by R_r the point set $-r<x<r$, $0\leqq y\leqq h$, and by Γ the unbounded region R_∞. Let E be a closed set on the boundary $y=0$ of R_r, and let CE be the set complementary to E with reference to the boundary $y=0$ of Γ. A solution interior to Γ of the parabolic equation

$$(1) \qquad F(u)\equiv\frac{\partial^2 u}{\partial x^2}-a(x, y)\frac{\partial u}{\partial x}-b(x, y)\frac{\partial u}{\partial y}-c(x, y)u=0,$$

$b>0$, will be termed a regular solution in $\Gamma-E$ if it is of class C^0 in $\Gamma-E$ and of class C^2 interior to Γ. The functions a, b, c in (1) are assumed continuous in Γ. The present paper is concerned mainly with uniqueness properties of solutions of (1) regular in $\Gamma-E$ under the assumption that there exists a positive function $K(x, y)$ with the properties: (a) $K(x, y)$ belongs to C^0 in $\Gamma-E$ and to class C^2 interior to Γ, and (b) $F(K)\leqq 0$. Such a $K(x, y)$ is shown to exist in the special case $a(x, y)=b(x, y)=0$, $c(x, y)+Ax^2+B>0$ (A and B positive constants) and the set E has measure zero. The following theorem is typical of the results obtained. The only solution $u(x, y)$ of (1) regular in $\Gamma-E$ with the properties $u(x, 0)=0$ with x in CE, $\lim u(x, y)/K(x, y)=0$ for point (x, y) approaching a point of E or for $|x|\to\infty$, is $u(x, y)\equiv 0$.

F. G. Dressel (Durham, N. C.).

Haack, Wolfgang, und Hellwig, Günter. Über Systeme hyperbolischer Differentialgleichungen erster Ordnung. I. Math. Z. 53, 244–266 (1950).

Haack, Wolfgang, und Hellwig, Günter. Über Systeme hyperbolischer Differentialgleichungen erster Ordnung. II. Math. Z. 53, 340–356 (1950).

The systems considered are of the form

$$\sum_{k=1}^{2}(a^k u_k+b^k v_k)+cu+dv+e=0,$$

$$\sum_{k=1}^{2}(\bar{a}^k u_k+\bar{b}_k v_k)+\bar{c}u+\bar{d}v+\bar{e}=0,$$

where $u_k=\partial u/\partial x_k$, $v_k=\partial v/\partial x_k$, $k=1, 2$. The bulk of the paper is concerned with the linear case, in which the coefficients are given functions of x_1, x_2. Assuming hyperbolic character, the system is reduced to canonical form. Existence of a solution of the Cauchy problem for a noncharacteristic initial line is proved by iteration under the assumptions that a^k, $\bar{a}^k$, b^k, $\bar{b}^k$ are of class C^2 and c, $\bar{c}$, d, $\bar{d}$, e, $\bar{e}$, and the data are of class C^1. Characteristic and mixed boundary value problems are solved and the solution of the Cauchy problem reduced to the determination of a generalized Riemann function. The work of the authors differs from previous treatments of the same problem mainly in the consistent use of integral theorems for Pfaffian forms. The authors also derive a canonical form of the equations in the quasi-linear case, and indicate their solution by difference methods. The results are applied to the equations for stationary flow of an ideal gas in the plane and to flow with cylindrical symmetry. *F. John* (Los Angeles, Calif.).

Conti, Roberto. Sul problema di Cauchy per le equazion di tipo misto $y^k z_{xx}-x^k z_{yy}=0$. II. Ann. Scuola Norm. Super. Pisa (3) 4, 1–25 (1950).

A solution z of $y^k z_{xx}-x^k z_{yy}=0$ satisfying $z(x, 0)=\theta(x)$, $z_y(x, 0)=\zeta(x)$ is to be found, where k is a positive integer. In the first part of this paper [Ann. Scuola Norm. Super. Pisa (3) 2 (1948), 105–130 (1950); these Rev. 11, 668] the author established the existence of a solution which is regular in the octant $0\leqq y\leqq x$ with the exception of the origin, under the assumptions that θ and ζ are continuous for $x\geqq 0$ and twice continuously differentiable for $x>0$. The second part is concerned with assumptions on the initial data that will also ensure regularity of z at the origin. It is found that the additional assumption that $\theta''(x)$ vanishes with x of order k and $\zeta''(x)$ of order $k-1$ is sufficient to guarantee that the first and second derivatives of z approach limits, as (x, y) tends to the origin through points of the octant. If k is even, and if $\zeta(x)$ and $\theta(x)$ are prescribed for all real x, the author finds that a regular solution exists for all (x, y), provided θ and ζ have continuous third derivatives, vanishing respectively of orders $k-1$ and $k-2$ for $x=0$, and provided $\theta''(0)=\zeta''(0)=0$. The continuation of z from the first octant into the whole plane is achieved by determining a solution z of the differential equation in $|y|\leqq x$ from its values on the characteristics $y=\pm x$ for $x\geqq 0$. *F. John.*

Ladyženskaya, O. On the method of Fourier for the wave equation. Doklady Akad. Nauk SSSR (N.S.) **75**, 765–768 (1950). (Russian)

Let Ω be a domain in the $(x_1, \cdots, x_n)$-space, u_k $(k=1, 2, \cdots)$ the normalized eigenfunction of the differential equation $\Delta u+\lambda u=0$ belonging to the domain Ω, the boundary condition $u=0$ and the eigenvalue λ_k. The series (1) $u(x_1, \cdots, x_n, t)=\sum(a_k \cos \lambda_k t+b_k \sin \lambda_k t)u_k$ is the formal solution of the wave equation $\Delta u=u_{tt}$ for the initial conditions $u=\phi_0$, $u_t=\phi_1$ and the boundary condition $u=0$. The author proves that the following conditions are sufficient for the uniform convergence of (1) and the series obtained from (1) by two term-by-term differentiations. The boundary surface S of Ω is a Lyapunov surface locally representable by $([\frac{1}{2}n]+4)$-times continuously differentiable functions. The eigenfunctions u_k have on $\Omega+S$ continuous derivatives of order $([\frac{1}{2}n]+3)$ and ϕ_j has continuous derivatives of order $([\frac{1}{2}n]+3)$ for $j=0$, $([\frac{1}{2}n]+2)$ for $j=1$. Moreover, ϕ_j and $\Delta^v\phi_j=0$ on S for $1\leqq v\leqq\frac{1}{2}([\frac{1}{2}n]+2)$ for $j=0$, for $1\leqq v\leqq\frac{1}{2}([\frac{1}{2}n]+1)$ for $j=1$. Similar results are stated for the boundary condition $(\partial u/\partial n)+hu=0$ on S. *L. Bers.*

Aleksandryan, R. A. On a problem of Sobolev for a special equation with partial derivatives of the fourth order. Doklady Akad. Nauk SSSR (N.S.) **73**, 631–634 (1950). (Russian)

Let D be a finite region of the (x, y)-plane, Γ its boundary. The problem is the existence of solutions of $\partial^2\Delta u/\partial t^2+w^2\partial^2u/\partial y^2=0$, where $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$, with $u|_\Gamma=0$ for $t\geqq 0$, $u|_{t=0}=\psi_0(x, y)$, $\partial u/\partial t|_{t=0}=\psi_1(x, y)$. The solution depends on the following construction. Let $\mathbf{V}=\mathbf{V}(x, y, t)$ be a vector, with components V_x, V_y such that div $\mathbf{V}=0$. Define the scalar $P=P(x, y, t)$ by $\Delta P=\rho w^2\partial V_x/\partial x$, with $P=0$ on Γ. Define the operator A so that the vector $A\mathbf{V}$ has components $(-w^2V_x+\rho^{-1}\partial P/\partial x,\ \rho^{-1}\partial P/\partial y)$. Let $\mathbf{V}$ satisfy the differential equation $\partial^2\mathbf{V}/\partial t^2=A\mathbf{V}$. Then P satisfies the original equation. By this means are established theorems of existence, uniqueness, and continuous dependence on the initial data. Also examined are the properties of the operator A in a certain Hilbert space.

F. V. Atkinson (Ibadan).

Aleksandryan, R. A. On Dirichlet's problem for the equation of a chord and on the completeness of a system of functions on the circle. Doklady Akad. Nauk SSSR (N.S.) **73**, 869–872 (1950). (Russian)

The author discusses the eigen-value problem in which u is to satisfy $\partial^2u/\partial x^2-\lambda^2\partial^2u/\partial y^2=0$ in the unit circle and is to vanish on the boundary of that circle. His theorem 1 states that λ_0 is an eigen-value if and only if arc tan λ_0/π is rational. Theorem 2 asserts the completeness of a set of the eigen-functions, which are expressible in terms of Čebyšev polynomials. Theorems 3 and 4 give properties relating to approximation and almost periodicity of the operator A and vector $\mathbf{V}$ [see the preceding review]. *F. V. Atkinson.*

***Krylov, A. N. O nekotoryh differencial'nyh uravneniyah matematičeskoĭ fiziki imeyuščih priloženie v tehničeskih voprosah. [On Some Differential Equations of Mathematical Physics Having Application to Technical Problems].** 5th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 368 pp.

The 4th edition of this book appeared in 1948 as part of Krylov's collected works. The first edition appeared in 1913, the 2d in 1932; there have been relatively few changes since the 2d.

Functional Analysis, Ergodic Theory

Grothendieck, Alexandre. Critères généraux de compacité dans les espaces vectoriels localement convexes. Pathologie des espaces ($\mathfrak{LF}$). C. R. Acad. Sci. Paris **231**, 940–941 (1950).

Given a locally convex linear T_2-space E, let E' be the dual space of all continuous linear functionals on E. A weak topology can then be imposed on E' with generic neighborhood of the origin defined in the usual way by a finite subset F of E as $[x'; |x'(x)|<1, x\epsilon F]$. One can now obtain a new convex neighborhood topology $\tau(E, E')$ on E with generic neighborhood of the origin defined by a compact subset K' of E' as $[x; |x'(x)|<1, x'\epsilon K']$. The principal theorem stated in this note is an extension of results due to M. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.), **14**, 5–7 (1937)] and to W. F. Eberlein [Proc. Nat. Acad. Sci. U. S. A. **33**, 51–53 (1947); these Rev. **9**, 42; **10**, 855], namely: Let A be a subset of E. In order that the convex closed extension C of A be compact it is necessary and sufficient that A be conditionally sequentially compact and that C be complete in the $\tau(E, E')$ topology. The note concludes with an announcement of some counter-examples which resolve problems in ($\mathfrak{L}$-$\mathfrak{F}$) spaces. *R. Phillips* (Princeton, N. J.).

Aczél, J. On quasi-linear functional operations. Publ. Math. Debrecen **1**, 248–250 (1950).

Let F be a real linear vector space with a topology. A real-valued continuous function on F is said to be quasilinear if $U(af+g)=U(af^*+g)$ whenever $U(f)=U(f^*)$. Such a function is said to be quasidistributive if $U(f+g)=U(f^*+g)$ whenever $U(f)=U(f^*)$. The author shows that the following are equivalent: (1) $U(f)$ is quasilinear; (2) $U(f)$ is quasidistributive; and (3) there exists a continuous, strictly monotone function $t(u)$ and a continuous linear functional $L(f)$ in F such that $U(f)=t[L(f)]$. *R. Phillips.*

Köthe, Gottfried. Neubegründung der Theorie der vollkommenen Räume. Math. Nachr. **4**, 70–80 (1951).

A perfect coordinate space (vollkommen Koordinatenraum) as defined by the author and O. Toeplitz [J. Reine Angew. Math. **171**, 193–226 (1934)] is a vector space λ whose elements are sequences of complex numbers and which has the property that $\lambda^{**}=\lambda$, where λ^* is the set of all sequences $u_1, u_2, \cdots$ such that $\sum_{i=1}^\infty|x_iu_i|<\infty$ for all $x_1, x_2, \cdots$ in λ. In the present paper a part of the theory of perfect coordinate spaces is presented anew in such a manner as to simplify it and make its connection with the theory of convex topological vector spaces clear and explicit. In addition a number of new results are proved. Five different topologies in λ are studied and most of the new results deal with characterizations of these topologies and with their relationship to one another. Examples: Theorem 17. The T_s topology is the strongest convex topology in λ for which convergence coincides with weak convergence. Theorem 19e. The topologies T_b and T_b in λ are identical if and only if the topologies T_s and T_s in λ^* are identical.

G. W. Mackey (Cambridge, Mass.).

Wilansky, Albert. The bounded additive operation on Banach space. Proc. Amer. Math. Soc. **2**, 46 (1951).

By adapting an argument due to Cauchy and Hamel, the author proves that an additive operation from one normed linear space to another is continuous (and hence homogeneous) if and only if it is bounded on the interior of a sphere. An example (one-dimensional) shows that, unlike

the homogeneous case, additivity and boundedness on the surface of the unit sphere does not imply continuity.
M. M. Day (Urbana, Ill.).

Yamabe, Hidehiko. On an extension of the Helly's theorem. Osaka Math. J. 2, 15–17 (1950).

The author establishes the following extension of Helly's theorem. Given a dense convex set M in a normed linear space E (not necessarily complete) and arbitrary x in E, $\epsilon>0$, and linear functionals $f_1, \cdots, f_n$, there exists a y in M such that $\|x-y\|<\epsilon$ and $f_i(x)=f_i(y)$ $(i=1, \cdots, n)$. When E is the space of continuous functions on $(0, 1)$ and M the subspace of polynomials the result yields a refinement of the Weierstrass approximation theorem. *W. F. Eberlein.*

James, Robert C. Bases and reflexivity of Banach spaces. Ann. of Math. (2) **52**, 518–527 (1950).

Various writers have conjectured that a Banach space B is reflexive if B is isomorphic with B^{**} or B^{**} is separable. If B^{**} is separable and B has an absolute basis, it is known that B is reflexive [cf. A. M. Gleason, Bull. Amer. Math. Soc. **53**, 52 (1947); S. Karlin, Duke Math. J. **15**, 971–985 (1948); these Rev. **10**, 548]. The author constructs a separable, nonreflexive Banach space B which is isomorphic with B^{**}. The method rests on a characterization of reflexive Banach spaces possessing a basis. If the basis is unconditional, B is reflexive if and only if it contains no subspace isomorphic with either c_0 or l^1. The Gleason-Karlin theorem is obtained as a corollary. *W. F. Eberlein.*

Mourier, Édith. Lois des grands nombres et théorie ergodique. C. R. Acad. Sci. Paris **232**, 923–925 (1951).

Let $\mathfrak{X}$ be a separable Banach space, and let $L^p(\mathfrak{X})$ be the L^p-space of all $\mathfrak{X}$-valued random variables $X(\omega)$ (defined on a probability space $\Omega=\{\omega\}$) such that $\int_\Omega\|X(\omega)\|^p d\omega<\infty$. Let further $\{X_n(\omega)\,|\,n=1, 2, \cdots\}$ be an independent sequence of random variables from $L^p(\mathfrak{X})$ with a common distribution. The author announces the following two theorems: (1) If $p=1$, then there exists a random variable $Y(\omega)\epsilon L^1(\mathfrak{X})$ such that $\lim_{n\to\infty}\|n^{-1}\sum_{i=1}^n X_i(\omega)-Y(\omega)\|=0$ for almost all ω; (2) if $p>1$ and if $\mathfrak{X}$ is reflexive, then there exists a random variable $Y(\omega)\epsilon L^p(\mathfrak{X})$ such that

$$\lim_{n\to\infty}\int_\Omega\|n^{-1}\sum_{i=1}^n X_i(\omega)-Y(\omega)\|^p d\omega=0.$$

S. Kakutani (New Haven, Conn.).

Arens, Richard. Representation of functionals by integrals. Duke Math. J. **17**, 499–506 (1950).

Premier théorème: Soit Q une famille de fonctions réelles définies sur un espace topologique X; supposons réalisées les conditions suivantes: (a) Chaque $f\epsilon Q$ est continue et a support compact; (b) pour chaque $f\epsilon Q$ il existe $g\epsilon Q$ telle que $f(x)\neq 0$ implique $g(x)\geqq 1$; (c) quelles que soient $f_1, \cdots, f_n\epsilon Q$ on a $\theta(f_1, \cdots, f_n)\epsilon Q$ pour toute fonction θ de n variables réelles, continue, et nulle en $(0, \cdots, 0)$; alors si $J(f)$ est une forme linéaire positive sur Q, on a $J(f)=\int f(x)dm(x)$ où m est une fonction additive d'ensembles sur X vérifiant la condition suivante: Pour tout ensemble mesurable et de mesure finie E, et tout $\epsilon>0$, il existe un ouvert $G\supset E$, réunion d'une suite de compacts, tel que $m(G)<m(E)+\epsilon$. Second théorème: Soit Q une famille de fonctions réelles sur X; supposons que: (a) Toute $f\epsilon Q$ est continue et tend vers 0 à l'infini; (b) Q est une algèbre, et toute $f\epsilon Q$ est différence d'éléments positifs de Q; soit alors $J(f)$ une forme linéaire positive sur Q, telle que pour toute f positive de Q on ait $J(f)=\sup_u J(uf)$ $(u\epsilon Q, 0\leqq u\leqq 1)$; alors il existe sur X une mesure positive m telle que chaque $f\epsilon Q$ soit m-sommable, avec $J(f)=\int f(x)dm(x)$, et Q est dense dans les espaces L^p construits à partir de m et de l'ensemble des limites uniformes de fonctions de Q. L'auteur semble avoir été inspiré par la démonstration due à Krein du théorème de Plancherel dans les groupes abéliens (il est de fait que le second théorème implique celui de Plancherel); il serait intéréssant de savoir si, réciproquement, le théorème de Plancherel dans les algèbres commutatives [voir un article du rapporteur, Ann. of Math. (2) **53**, 68–124 (1951); ces Rev. **12**, 421] permet de retrouver le second théorème de l'auteur; c'est fort probable dans le cas où X est localement compact, cas auquel il doit être possible de se ramener.
R. Godement (Nancy).

Jerison, Meyer. Characterizations of certain spaces of continuous functions. Trans. Amer. Math. Soc. **70**, 103–113 (1951).

A real Banach space B is said to have property A if every collection Γ of maximal convex subsets of the surface of the unit sphere in B with empty intersection contains a pair of directed systems C_a', C_a'' such that $d(b, C')+d(b, C'')\to 2$ for all b with $\|b\|\leqq 1$ [see Arens and Kelley, same Trans. **62**, 499–508 (1947); these Rev. **9**, 291]. Here $d(b, C)$ means $\inf_{c\epsilon C}\|b-c\|$. The author, using the T-sets and F_T functionals of the reviewer [Ann. of Math. (2) **49**, 132–140 (1948); these Rev. **9**, 291] proves the following results. (1) B has property A if and only if there exists a compact space X and an involutory homeomorphism $\sigma(X)$ such that B is equivalent to the Banach space $B_\sigma(X)$ of real continuous functions on X for which $f(\sigma(x))=-f(x)$. Also, if X, X' are compact spaces with respective involutory homeomorphisms σ, σ', then, if $B_\sigma(X)$ and $B_{\sigma'}(X')$ are equivalent, X and X' are homeomorphic provided that σ, σ' both have no fixed points or exactly 1 fixed point. (2) B is equivalent to $B(X)$, the space of all real continuous functions on X, for some compact X, if and only if B has property A and the unit sphere of B has an extreme point. This eliminates one of the conditions used to characterize $B(X)$ by Arens and Kelley [loc. cit.]. (3) B is equivalent to the Banach space of all continuous functions vanishing at infinity on some locally compact (Hausdorff) space X if and only if B has property A and the extreme points of the unit sphere in B^* lie in two disjoint antipodal sets. *S. B. Myers.*

Sargent, W. L. C. On linear functionals in spaces of conditionally integrable functions. Quart. J. Math., Oxford Ser. (2) **1**, 288–298 (1950).

The principal results of this paper state that the uniform boundedness theorem and the Banach-Steinhaus theorem hold for sequences of linear functionals on the incomplete normed linear space (I) of Denjoy integrable functions on $[0, 1]$ with norm $\|x\|=\sup_{0\leq t\leq 1}|\int_0^t x(u)du|$. The author investigates properties of the integral representations of such functionals and shows also that a sequence $\{F_n\}$ of linear functionals on $C_0(0, 1)$ is bounded in norm if and only if $\limsup_n|F_n(y)|<\infty$ for each y in the first category subspace CI of C_0 consisting of functions of the form $y(t)=\int_0^t x(u)du$, $x\epsilon(I)$. *R. E. Fullerton* (Madison, Wis.).

Walters, Stanley S. The space H^p with $0<p<1$. Proc. Amer. Math. Soc. **1**, 800–805 (1950).

Let H^p be the space of those functions analytic in $|z|<1$ such that $|f|=\sup_{0\leq r<1}(\int_0^{2\pi}|f(re^{i\theta})|^p d\theta)^{1/p}<\infty$. For $0<p<1$, H^p is equivalent to a subspace of $L^p(0, 2\pi)$, a space shown

by the reviewer to have no nonzero linear functionals. This paper shows that the functionals $\gamma_{n,z}(f)=f^{(n)}(z)/n!$ are linear on H^p and gives bounds for the norms of the $\gamma_{n,z}$. Since H^{p*} is total over H^p, weak convergence in H^p is meaningful. It is shown that weak convergence of a sequence in H^p implies uniform convergence on every compact subset of $|z|<1$. *M. M. Day* (Urbana, Ill).

Checcucci, Vittorio. Sui fondamenti del calcolo con matrici infinite. Ann. Scuola Norm. Super. Pisa (3) 4, 205–222 (1950).

This paper is an exposition of elementary properties, mostly well-known, of infinite matrices $A\equiv(a_{ij})$ of type (p,q), i.e., where the rows are of l^p $(\sum_i|x_i|^p$ convergent, $p\geqq 1)$ and columns of l^q, with p or $q=\infty$ (bounded sequences) permitted. The main objective is to show that limitedness of the matrices involved is sufficient for the associative property of the product: $A(BC)=(AB)C$.

T. H. Hildebrandt (Ann Arbor, Mich.).

Phillips, R. S. On one-parameter semi-groups of linear transformations. Proc. Amer. Math. Soc. 2, 234–237 (1951).

The author proves that if $T(\xi)$, $\xi>0$, is a strongly measurable semi-group of linear bounded operations on a B-space X to itself, then $\|T(\xi)\|$ is bounded in each interval $[\delta, 1/\delta]$. The importance of the result lies in the fact that it shows that strong measurability alone suffices to make $T(\xi)$ strongly continuous for $\xi>0$. [Cf. the reviewer's Functional Analysis and Semi-Groups [Amer. Math. Soc. Colloquium Publ., v. 31, New York, 1948; these Rev. 9, 594], where the continuity theorem is proved under the assumption that $T(\xi)$ is strongly measurable and $\|T(\xi)\|$ is bounded in each interval $[\delta, 1/\delta]$. Dunford [Ann. of Math. (2) 39, 569–573 (1938)] had previously proved right hand continuity from strong measurability and boundedness of the norm in some interval $(0,a)$.] The author shows by a counterexample that strong measurability cannot be replaced by weak measurability in his theorem. *E. Hille.*

Dixmier, J. L'adjoint du produit de deux opérateurs fermés. Ann. Fac. Sci. Univ. Toulouse (4) 11 (1947), 101–106 (1949).

This paper considers a generalized product of two distributive, possibly multiple-valued, operators on a Hilbert space H to H. Write $D(A)$, $R(A)$, $\bar{A}$, and A^* for the domain, range, closed linear extension, and adjoint of A. Then $x\epsilon D_{A\cdot B}$ and $y=A\cdot Bx$ if sequences $\{x^n\}\subset D(A)$ and $\{y^n\}\subset R(A)$ exist such that (1) $\lim_{n\to\infty}(\inf_{s^n\epsilon A^{-1}y^n,\, w^n\epsilon Bx^n}\|s^n-w^n\|)=0$, where the formulation of (1) permits A^{-1} and B to be multiply-valued. The main results are (2) $A\cdot B=\widetilde{A\cdot B}\supset\bar{A}\cdot\bar{B}$ and (3) $(A\cdot B)^*=B^*A^*\subset(AB)^*$. Examples are cited to show that unless conditions such as closure or continuity are imposed, the inclusions in (2) and (3) in the usual sense of extensions, cannot be replaced by equality assertions.

D. G. Bourgin (Urbana, Ill.).

Gavurin, M. K. On estimates for the characteristic numbers and vectors of a perturbed operator. Doklady Akad. Nauk SSSR (N.S.) 76, 769–770 (1951). (Russian)

Let A_0 be a bounded linear operator in a Hilbert space, λ_0 one of its characteristic values, and φ_0 a corresponding characteristic vector. Then, under certain restrictions, to each A in a certain neighbourhood of A_0 there is a characteristic value λ and a characteristic vector φ lying respectively in neighbourhoods of λ_0 and φ_0. The author's theorem includes bounds for $|\lambda-\lambda_0|$ and $\|\varphi-\varphi_0\|$ in terms of $\|A-A_0\|$ and $\|R\|$, where R is a generalised inverse of $A_0-\lambda_0$. Other results relate to the location of the rest of the spectrum of A, the analytic expression of φ as a function of λ and A, and corresponding results for the adjoint operators. The bounds for $|\lambda-\lambda_0|$ and $\|\varphi-\varphi_0\|$ are claimed to be incapable of improvement. No proofs are given. *F. V. Atkinson.*

Sherman, S. Non-negative observables are squares. Proc. Amer. Math. Soc. 2, 31–33 (1951).

It is shown that, in the terminology of the reviewer [Ann. of Math. (2) 48, 930–948 (1947); these Rev. 9, 241], every sum of squares of observables is itself the square of an observable. As the partial ordering used is precisely that in which the positive elements are the sums of squares, the author's result permits some results pertaining to the ordering to be sharpened. In particular it follows that for any spectral value of an observable, there is always a pure state of the system of all observables in which the expectation value of the given observable is the specified spectral value (previously this was known only for systems of observables consisting of the self-adjoint elements of an algebra of operators on Hilbert space [Segal, Bull. Amer. Math. Soc. 53, 73–88 (1947); these Rev. 8, 520]). *I. E. Segal.*

Daleckiĭ, Yu. L., and Kreĭn, S. G. Formulas of differentiation according to a parameter of functions of Hermitian operators. Doklady Akad. Nauk SSSR (N.S.) 76, 13–16 (1951). (Russian)

For each value of τ in (a,b) let $H(\tau)$ be a bounded Hermitian operator in Hilbert space. It is stated that if $f(\lambda)$ is a function of the real variable λ with an absolutely continuous derivative $df/d\lambda$ in some neighbourhood of the spectrum of $H(\tau_0)$, and $E_\lambda(\tau_0)$ is the spectral set of $H(\tau_0)$, then

$$\frac{df(H(\tau_0))}{d\tau}=\int\int\frac{f(\lambda)-f(\mu)}{\lambda-\mu}dE_\lambda(\tau_0)\frac{dH(\tau_0)}{d\tau}dE_\mu(\tau_0),$$

where the integral is interpreted as an abstract repeated Stieltjes integral. Analogous formulae are given for derivatives of the form $df(H(\tau_0),\tau)/d\tau$ and for higher order derivatives of $f(H(\tau))$. The latter are applied to give the expansion of $f(H_1+\epsilon H_2)$ in powers of ϵ. Further applications are to solution of the equation $H(\tau)x(\tau)=g(\tau)$, for the unknown element $x(\tau)$ of the space, and of the equation $H(\tau)X(\tau)-X(\tau)H(\tau)=F(\tau)$ for the unknown operator $X(\tau)$.

J. L. B. Cooper (Cardiff).

Krasnosel'skiĭ, M. A. Operators with monotonic minorants. Doklady Akad. Nauk SSSR (N.S.) 76, 481–484 (1951). (Russian)

Kreĭn and Rutman [Uspehi Matem. Nauk (N.S.) 3, no. 1(23), 3–95 (1948) = Amer. Math. Soc. Translation no. 26 (1950); these Rev. 10, 256; 12, 341] have established existence theorems for characteristic vectors of certain (linear and nonlinear) operators A in a Banach space E that are positive and monotone with respect to the partial ordering $\geqq$ introduced in E by a fixed cone $K<E$. In this paper a few similar results concerning nonmonotone operators are communicated. The completely continuous operator B is said to have a monotone minorant A if $Bx\geqq Ax$ for all $x\epsilon K$, where A is an operator as mentioned above. It is stated that such an operator B has characteristic vectors of arbitrary norm in the cone K, moreover these characteristic vectors form a continuous branch F, i.e. F has a nonvoid intersection with the boundary of every open set containing

the vertex of K. Various applications to the characteristic value problem for nonlinear integral equations are indicated.
M. Golomb (Lafayette, Ind.).

Pierce, R. Cones and the decomposition of functionals. Math. Mag. **24**, 117–122 (1951).

Let E be a partially ordered linear space with unit. The author has found an independent proof for the canonical decomposition of a linear functional into the difference of nonnegative functionals. The original proof of a slightly more general theorem was given by J. Grosberg and M. Kreĭn [C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 723–726 (1939); these Rev. **1**, 338]. *R. Phillips*.

Šilov, G. E. On continuous sums of finite-dimensional rings. Mat. Sbornik N.S. **27(69)**, 471–484 (1950). (Russian)

The author has proved elsewhere [Trav. Inst. Math. Stekloff **21** (1947); these Rev. **9**, 596] that a normed ring of "type C" with one generator [see the review of the cited paper for definitions] can be represented as a "continuous direct sum" of primary rings K_t with one generator, where t ranges over a compact subset of the complex plane. Such continuous direct sums need not be of type C; therefore a natural problem here is to give conditions under which a continuous direct sum is of type C. This problem was solved in the paper cited above for the case in which each K_t is equal to the same ring of power series. It has also been solved by the author [Mat. Sbornik N.S. **26(68)**, 291–310 (1950); these Rev. **11**, 602] for the case in which each K_t ($a\leqq t\leqq b$) has dimension at most equal to 2. In this case the problem reduces to that of determining all normed rings of continuous functions on $a\leqq t\leqq b$ which contain all functions with continuous first derivatives. In the present paper a sufficient condition (too complicated to be stated here) is obtained for the case in which each K_t has dimension at most $N+1$, where N is independent of t and $a\leqq t\leqq b$. When $N=1$, the condition is also necessary and reduces to that given in the second paper mentioned above. Whether or not the condition is necessary in the general case remains open.
C. E. Rickart (New Haven, Conn.).

Mil'man, D. On the theory of rings with involution. Doklady Akad. Nauk SSSR (N.S.) **76**, 349–352 (1951). (Russian)

This paper contains a number of applications of extreme point theorems to rings with involution as considered by Naĭmark [Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 52–145 (1948) = Amer. Math. Soc. Translation no. **25** (1950); these Rev. **10**, 308; **12**, 111]. The ring R (with identity e and involution $x\rightarrow x^*$) is assumed to be "s-reduced" in the sense that $f(x_0)=0$, for every functional f positive on R ($f(x^*x)\geqq 0$ for all x), implies $x_0=0$. An s-reduced ring is reduced in the sense of Naĭmark [cf. above translation, p. 49] and a reduced symmetric ring is s-reduced. The ring R is also assumed to possess a regular norm (i.e. every positive functional on R can be extended to a positive functional on the completion of R under the given norm). The set K of all positive functionals on R such that $f(e)=1$ is a convex set in the space $\bar{H}$ conjugate to the space H of all Hermitian elements in R and is compact in the weak H-topology of $\bar{H}$. A homomorphism $x\rightarrow A_x$ of R into the ring of bounded operators on a Hilbert space $\mathfrak{H}$ (not necessarily separable or finite dimensional) is called a representation of R if $A_{x^*}=A_x^*$. A representation A_x is called "cyclic" if there exists a "cyclic vector" $\xi_0\epsilon\mathfrak{H}$ such that elements of the form $A_x\xi_0$, $x\epsilon R$, are dense in $\mathfrak{H}$. If $\|\xi_0\|=1$ and $f(x)=(A_x\xi_0, \xi_0)$, then $f\epsilon K$. Conversely, every $f\epsilon K$ can be obtained in this way and the corresponding cyclic representation is uniquely determined by f up to an equivalence [op. cit., §6]. An element $h\epsilon H$ is said to be "realized exactly" by a cyclic representation A_x with cyclic vector ξ_0, $\|\xi_0\|=1$, if ξ_0 is an eigenvector of A_h with eigenvalue either $\min_{f\epsilon K}f(h)$ or $\max_{f\epsilon K}f(h)$. The representation in this case is uniquely determined up to an equivalence if, and only if, the associated positive functional is an extremal point of K. Any subset of H, each of whose elements is realized exactly by the same representation, is contained in a maximal set with the same property (perhaps with a different representation) and every such maximal set has associated with it an irreducible representation. The set of all these irreducible representations is complete (i.e. if $x_0\neq 0$, then there exists one of the representations A_x such that $A_{x_0}\neq 0$).
C. E. Rickart (New Haven, Conn.).

Iwamura, Tsurane. On continuous geometries. II. J. Math. Soc. Japan **2**, 148–164 (1950).

Let L be a continuous geometry, in general reducible. A mapping of L into a conditionally complete lattice-group G: $\delta=\delta(x)$ is called a dimension function if each $\delta\geqq 0$ and (1) $\delta(x+y)=\delta(x)+\delta(y)$ if x, y are independent; (2) δ-equalities are unrestrictedly additive, $\delta(V^{\perp}x_\gamma)=\delta(V^{\perp}y_\gamma)$ if $\delta(x_\gamma)=\delta(y_\gamma)$ for each γ ($^{\perp}$ indicates independent set of elements); (3) $\delta(a)<\delta(b)$ implies the existence of an $x<b$ with $\delta(x)=\delta(a)$; (4) $0<f\epsilon G$ implies $0<\delta(x)\leqq f$ for some $x\epsilon L$; (5) $\delta(x)=0$ only if $x=0$. Then each such δ is necessarily unrestrictedly additive, that is, $\delta(V^{\perp}x_\gamma)=\sum\delta(x_\gamma)$. A special such dimension $\delta_0(x)$ was constructed in part I [Jap. J. Math. **19**, 57–71 (1944); these Rev. **8**, 35] with the additional property: $\delta_0(x)=\delta_0(y)$ if and only if x, y are perspective. Thus the author's previous proof that perspectivity is unrestrictedly additive is simplified. Unlike the reviewer's direct proof [Duke Math. J. **5**, 503–511 (1939); these Rev. **1**, 30], it still requires the von Neumann dimension theory and his analysis of the centre. The author then considers a congruence relation $x\sim y$ which includes perspectivity (x, y perspective implies $x\sim y$) and for which: $(V^{\perp}x_\gamma)\sim(V^{\perp}y_\gamma)$ if $x_\gamma\sim y_\gamma$ for each γ; $x\sim(V^{\perp}y_\gamma)$ implies a decomposition $x=V^{\perp}x_\gamma$ with $x_\gamma\sim y_\gamma$; and $x\sim x_1\leqq x$ implies $x=x_1$. Every dimension function $\delta(x)$ defines such a relation by: $x\sim y$ if $\delta(x)=\delta(y)$. Conversely, every such relation can be so obtained from a dimension function. This is shown by generalizing the results of part I (construction of the dimension function and decomposition of the reducible geometry into a subdirect sum of irreducible components) to the present situation with $\sim$ in place of perspectivity. *I. Halperin*.

Theory of Probability

***Borel, Émile. Probabilité et certitude.** Presses Universitaires de France, Paris, 1950. 136 pp.

This booklet consists of a nontechnical discussion of the foundations of probability theory. Its main purpose appears to be to make the semantic point that "events of very small probability are practically impossible." *P. R. Halmos*.

Matusita, Kameo. On the fundamental operations of collectives. Ann. Inst. Statist. Math., Tokyo **2**, 5–11 (1950).

This paper is a study of the interaction of the four operations on collectives defined by von Mises. Each operation transforms a given collective. It also induces a transformation of the set of selections associated with that collective. Thus, to prove the existence of a collective it is necessary to prove the existence of all of its transforms. The author uses the results of Wald [Ergebn. Math. Kolloq. Wien **8**, 38–72 (1937)] in answering this question. *A. H. Copeland, Sr.*

Baticle, Edgar. Sur la probabilité des itérations dans le schéma de Bernoulli. C. R. Acad. Sci. Paris **232**, 472–473 (1951).

Let n ($<a+b$) balls be drawn successively from an urn containing a white and b black balls. The author calculates explicitly the probability that there are exactly k runs of white balls of length not less than x. *K. L. Chung.*

Bass, Jean. Sur la compatibilité des lois de probabilité. C. R. Acad. Sci. Paris **231**, 755–756 (1950).

Associated with three random variables are the joint distribution of all three, the three joint distributions of the pairs, and the three individual distributions. The author studies certain necessary conditions for the compatability of these distributions. *A. H. Copeland, Sr.*

Bass, Jean. Étude géométrique du problème de la compatibilité des lois de probabilités. C. R. Acad. Sci. Paris **232**, 593–595 (1951).

This is the second of two notes [see the preceding review] concerning the compatibility of the individual and joint distributions associated with three random variables. The author uses the method of Karhunen to obtain compatibility conditions for the case of discrete variables.
A. H. Copeland, Sr. (Ann Arbor, Mich.).

Steinhaus, H. The so-called Petersburg paradox. Colloquium Math. **2**, 56–58 (1949).

Construct a sequence $\{a_n\}$ by first letting ones alternate with empty places, then fill every second empty place by a two, next every second remaining empty space by a four, etc. If the entrance fee at the nth trial of a St. Petersburg game is fixed at a_n, then there is probability one that the sequence of actual gains will have the same distribution function as $\{a_n\}$. *W. Feller* (Princeton, N. J.).

Rényi, Alfréd. Contributions to the theory of independent random variables. Acta Math. Acad. Sci. Hungar. **1**, 99–108 (1950). (Russian. English summary)

A sequence of real-valued functions defined, for instance, on the unit interval is called maximal if it separates almost all points of the interval. Suppose that $\{f_n\}$ is a maximal sequence of independent random variables, such that each f_n assumes only a finite number of different values, that the variance of $\sum_{j=1}^{n} f_j$ tends to infinity with n, and that A is a measurable set of positive measure. Let $\{g_n\}$ be the sequence of partial sums of the f's, normalized in the usual manner, so that the mean and the variance of each g_n are 0 and 1, respectively. The author's main result (theorem 1) is that if the distribution functions of the g's converge to the (necessarily normalized) Gaussian distribution, then the conditional distributions of the g's, relative to A, do the same thing. Theorems 2, 3, and 4 are easy consequences of theorem 1, and are all of the same form: From an assumed convergence of distribution functions one concludes to another such convergence. *P. R. Halmos.*

Andersen, Erik Sparre. On the frequency of positive partial sums of a series of random variables. Mat. Tidsskr. B. **1950**, 33–35 (1950).

This is a continuation of the author's previous investigation [Skand. Actuarietidskr. **32**, 27–36 (1949); these Rev. **11**, 256]. Several results are stated without proof. The following are typical: Let $X_1, X_2, \cdots$ be independent random variables, each having the same distribution function $F(x)$ and let N_n denote the number of sums among $X_1, X_1+X_2, \cdots, X_1+\cdots+X_n$ which are positive. Then $\Pr\{N_n=m\}=\Pr\{N_m=m\}\cdot\Pr\{N_{n-m}=0\}$. If in addition to the assumption stated above we have

$$\Pr\{X_1+\cdots+X_k>0\}=a, \quad k=1, 2, \cdots,$$

then $\lim_{n\to\infty}\Pr\{N_n/n<\alpha\}=\sin \pi a/\pi\int_0^{\alpha}[x^{1-a}(1-x)^a]^{-1}dx$. The last result is an extension of the "arc sin law" which the author discussed in the paper cited above. *M. Kac.*

Gnedenko, B. V. Limit theorems for sums of independent random variables. Amer. Math. Soc. Translation no. **45**, 82 pp. (1951).

Translated from Uspehi Matem. Nauk **10**, 115–165 (1944); these Rev. **7**, 19.

Mattila, Sakari. On the weak topology in the theory of probability. Soc. Sci. Fenn. Comment. Phys.-Math. **15**, no. 14, 44 pp. (1950).

Let $F'=\{x'(\omega)\}$, $F=\{x(\omega)\}$ be two linear families of random variables defined on the same probability space $\Omega=\{\omega\}$. Assume that the "mixed moment"

$$E(x', x)=\int_\Omega x'(\omega)x(\omega)d\omega$$

exists for any $x'\epsilon F'$, $x\epsilon F$, and satisfies the following conditions: (1) $E(x', x)=0$ for all $x'\epsilon F'$ if and only if $x(\omega)=0$ for almost all ω; (2) $E(x', x)=0$ for all $x\epsilon F$ if and only if $x'(\omega)=0$ for almost all ω. The pair (F', F) may be considered as a dual linear system in the sense of Mackey [Trans. Amer. Math. Soc. **57**, 155–207 (1945); these Rev. **6**, 274; **7**, 620] and is called a conjugate pair of random spaces. This mixed moment $E(x', x)$ makes it possible to introduce weak topologies on F' and F. Let now $S=\{s\}$ be a topological space or a measure space. An F-valued function $x_s=x_s(\omega)$ defined on S is called weakly continuous or weakly measurable if $E(x', x_s)$ is a continuous function or a measurable function of s for any $x'\epsilon F'$. Further, x_s is weakly integrable if, for any measurable subset A of S, there exists an $x_A\epsilon F$ such that $\int_A E(x', x_s)ds=E(x', x_A)$ for any $x'\epsilon F'$. The author shows that it is possible to apply this theory of weak integration to generalize the theory of stochastic processes recently developed by Cramér [Ark. Mat. Astr. Fys. **28B**, no. 12 (1942); these Rev. **4**, 13], Karhunen [Ann. Acad. Sci. Fennicae Ser. A. I. Math.-Phys. no. **34** (1946); no. **37** (1947); these Rev. **9**, 292], and Loève [Appendix to P. Lévy, Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1948; these Rev. **10**, 551] from the case when both F', F are L^2-spaces to the case when (F', F) is any conjugate pair of random spaces. *S. Kakutani.*

Lévy, Paul. Deux nouveaux exemples de processus stochastiques. C. R. Acad. Sci. Paris **231**, 1208–1210 (1950).

Two examples of curious possibilities for Markov processes with denumerably many states E_k are given. To each

E_k there corresponds a mean sojourn time μ_k. If $\mu_k = \infty$ the state is final, if $\mu_k = 0$ it is ephemeral. In the first example the E_k stand in one-to-one correspondence with the rationals r_k in $(-\infty, \infty)$; the system necessarily passes through the states in their natural order and the sum over μ_k with $a < r_k < b$ is finite but increases as $b \to \infty$ or $a \to -\infty$. In the second example there is an ephemeral state out of which the system succeeds in coming. A detailed description is to be given elsewhere. *W. Feller* (Princeton, N. J.).

Foster, F. G. Markoff chains with an enumerable number of states and a class of cascade processes. Proc. Cambridge Philos. Soc. **47**, 77–85 (1951).

The time average $\pi_{ij} = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} p_{ij}^{(k)}$ exists for a Markoff chain $A = (p_{ij})$ with an enumerable number of states [Kolmogoroff, Rec. Math. [Mat. Sbornik] N.S. **1**(43), 607–610 (1936); Yosida and Kakutani, Jap. J. Math. **16**, 47–55 (1939); these Rev. **1**, 62]. The chain A is called dissipative if $\pi_{ij} = 0$ for all i, j; nondissipative if $\sum_{j=1}^{\infty} \pi_{ij} = 1$ for all i. The author discusses the conditions for the nondissipative case: (i) A is dissipative if and only if A has no stable distribution x, i.e., $x = (x_1, x_2, \cdots)$ with $x_i \geqq 0$, $\sum_{i=1}^{\infty} x_i = 1$, $\sum_{i=1}^{\infty} x_i p_{ij} = x_j$; (ii) A is nondissipative if $\lim_{N\to\infty} \sum_{j=1}^{N} p_{ij} = 1$ uniformly in i; (iii) A is nondissipative if, given j, $N(j)$ can be found such that $\sum_{j=1}^{N} p_{ij}^{(k)} \geqq \sum_{j=1}^{N} p_{ij}$ for all k, and $N \geqq N(i)$. The results are applied and refined to the so-called cascade process [I. J. Good, Proc. Cambridge Philos. Soc. **45**, 360–363 (1949); R. Otter, Ann. Math. Statistics **20**, 206–224 (1949); these Rev. **10**, 614; **11**, 41], considered as a special kind of Markoff chain. *K. Yosida* (Nagoya).

Blanc-Lapierre, André. Remarques sur les fonctions aléatoires stationnaires laplaciennes. C. R. Acad. Sci. Paris **232**, 934–936 (1951).

The author discusses a Gaussian stochastic process $\{X(t), -\infty < t < \infty\}$, where $X(t)$ is a k-dimensional vector random variable which can be written in the form $X(t) = \int_{-\infty}^{\infty} e^{2\pi i\nu t} dx(\nu)$. The $x(\nu) - x(0)$ process is then also Gaussian. The correlation properties of the $x(\nu)$ process are described in detail under the hypothesis that the $X(t)$ process is stationary. The element $dz(\nu, t) = e^{2\pi i\nu t} dx(\nu) + e^{-2\pi i\nu t} dx(-\nu)$ is the general monochromatic component of the above representation of $X(t)$. It is said to be nonpolarized if $E\{|dx(\nu)|^2\} = E\{|dx(-\nu)|^2\}$ and if $E\{dx(\nu)dx(-\nu)\} = 0$, and is said to be elliptically polarized if (with probability 1) the dz ellipses described when t varies have a common eccentricity and major axis orientation, and are described in the same sense. (The exact meaning of this criterion is not stated.) It is stated that there is elliptic polarization if and only if $|E\{dx(\nu)dx(-\nu)\}|^2 = E\{|dx(\nu)|^2\} E\{|dx(-\nu)|^2\}$. In general, for each ν, dz can be considered as the sum of two independent Gaussian components, one elliptically polarized, the other nonpolarized. [For similar work from a different point of view see Wiener, Acta Math. **55**, 117–258 (1930).] *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, André. Sur l'analyse harmonique des fonctions aléatoires stationnaires, laplaciennes. C. R. Acad. Sci. Paris **232**, 1070–1072 (1951).

Continuing his first paper on this subject [see the preceding review], the author defines a class of transformations of elliptic vibrations which, when applied to monochromatic components of a light wave [see the preceding review] conserve the stationarity and Gaussian character of the stochastic process under discussion. He then finds the distribution of the parameters of the polarization ellipse. Hurwitz [J. Opt. Soc. Amer. **35**, 525–531 (1945); these Rev. **7**, 98] found the corresponding results in the nonpolarized case. *J. L. Doob* (Urbana, Ill.).

Gnedenko, B. V. On the theory of growth of homogeneous random processes with independent increments. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1948**, no. 10, 60–82 (1948). (Ukrainian. Russian summary)

The principal result is the following: If $\xi(t)$ ($\xi(0) = 0$) is a time-homogeneous process with independent increments, then a necessary and sufficient condition for existence of a nondecreasing positive function $u(t)$ such that $\limsup_{t\to\infty} |\xi(t)|/u(t) = a$ is the existence of a number c_0, $\frac{1}{2}a < c_0 < 2a$, such that $\int_1^{\infty} \Pr\{|\xi(t)| > cu(t)\} t^{-1} dt$ diverges for $c < c_0$ and converges for $c > c_0$. If $\xi(t)$ is a "stable" process (non-Gaussian) then for every nondecreasing positive $u(t)$ $\limsup_{t\to\infty} |\xi(t)|/u(t)$ is either 0 or ∞. *M. Kac.*

Bartlett, M. S., and Kendall, David G. On the use of the characteristic functional in the analysis of some stochastic processes occurring in physics and biology. Proc. Cambridge Philos. Soc. **47**, 65–76 (1951).

In the usual birth and death process, at each time t let k be 1 or 0 according as the initial individual is alive or dead, let $x_1, x_2, \cdots$ be the ages of the other live individuals, and let θ be any function of a positive real variable. The authors evaluate $E\{w^k e^{i\Sigma\theta(x_j)}\}$, the characteristic functional of the process, assuming constant birth and death rates but allowing immigration (thus extending previous results). The authors also find an integral equation for this functional when the birth and death rates are age dependent. In the application to cosmic ray showers there are two types of individual: electrons and photons. A suitable characteristic functional is defined, and an integral equation for it is derived. The authors show how the process distributions can be obtained from a knowledge of the characteristic functional. *J. L. Doob* (Urbana, Ill.).

Udeschini, Paolo. Meccanica aleatoria. Rend. Sem. Mat. Fis. Milano **20** (1949), 54–80 (1950).

Expository paper on the subject of the title.

Grace, Walter L., and Nesbitt, Cecil J. On average age at death problems. Soc. Actuar. Trans. **21**, no. 2, 70–74 (1950).

The purpose of this note is to demonstrate a general method for the solution of "average age at death problems," a class of problems of an academic nature commonly given as exercises to actuarial students. *T. N. E. Greville.*

Mathematical Statistics

Singh, Kuldip. The central points and parameter of distribution. J. Univ. Bombay (N.S.) **19**, part 3, sect. A, 1–11 (1950).

This paper should be listed under Geometry.

Hartar, Harman Leon. On the distribution of Wald's classification statistic. Ann. Math. Statistics **22**, 58–67 (1951).

Let the variates $t_{i\beta}$ ($i = 1, \cdots, p$; $\beta = 1, \cdots, n+2$) be normally and independently distributed with unit variance and expected values $E(t_{i\alpha}) = 0$ ($\alpha = 1, \cdots, n$), $E(t_{i,n+1}) = \rho_i$, $E(t_{i,n+2}) = \zeta_i$ where ρ_i and ζ_i are constants and define $\|s^{ij}\| = \|s_{ij}\|^{-1}$, $s_{ij} = \sum_{\alpha=1}^{n} t_{i\alpha} t_{j\alpha}/n$. The paper investigates the

distribution of the statistic $V=\sum_{i=1}^{p}\sum_{j=1}^{p}s^{ij}t_{i,n+1}t_{j,n+2}$ suggested by Wald [same Ann. **15**, 145–162 (1944); these Rev. **6**, 9] in connection with a problem of classification. The author derives the exact distribution of V for $p=1$. For $p>1$, the distribution of a certain approximation to the statistic V already suggested by Wald is investigated for the case $\rho_i=\zeta_i=0$. An empirically found distribution is compared with theoretical results. *G. E. Noether.*

Massey, Frank J., Jr. The distribution of the maximum deviation between two sample cumulative step functions. Ann. Math. Statistics **22**, 125–128 (1951).

Let $D=\max_x|S_n(x)-S'_m(x)|$, where $S_n(x)$ and $S'_m(x)$ are respective distribution functions from two populations with continuous distribution functions. Smirnoff [Bull. Math. Univ. Moscou **2**, no. 2 (1939); these Rev. **1**, 345; see also Feller, Ann. Math. Statistics **19**, 177–189 (1948); these Rev. **9**, 599; **10**, 855] derived the limiting distribution function of $D[mn/(m+n)]^{\frac{1}{2}}$ and tabled it [ibid., 279–281 (1948); these Rev. **9**, 599]. In this note the author gives a simple method for computing the distribution function of D for small samples and tables it for $n=m\leqq 40$. *M. Loève.*

Dixon, W. J. Ratios involving extreme values. Ann. Math. Statistics **22**, 68–78 (1951).

Let $x_1, x_2, \cdots, x_n$ be the observations of a sample arranged so that $x_1<x_2<\cdots<x_n$. In testing for the contamination of the sample by a few observations from a population with a different mean value one may use the ratio $r=(x_n-x_{n-j})/(x_n-x_i)$ for some small values of i and j. The distribution of these ratios is obtained for the case where the population from which the sample is drawn is (1) rectangular and (2) normal with $n\leqq 30$. *H. Chernoff.*

Gumbel, E. J., and Greenwood, J. Arthur. Table of the asymptotic distribution of the second extreme. Ann. Math. Statistics **22**, 121–124 (1951).

A table is constructed for the asymptotic distribution of the second largest value of a set of observations from a population with continuous distribution of the exponential type. This table is based on a result of Gumbel [Ann. Inst. H. Poincaré **5**, 115–158 (1935)] which may be similarly applied to the rth largest observation, with the use of the table of the incomplete Gamma function. *H. Chernoff.*

Hald, A., and Sinkbæk, S. A. A table of percentage points of the χ^2-distribution. Skand. Aktuarietidskr. **33**, 168–175 (1950).

Percentage points for the chi-square distribution were computed for 21 values of P (the percentage) and for 38 values of f (degrees of freedom): $f=1(1)10(2)32(4)100$. These values were obtained by inverse interpolation in tables of the incomplete Gamma function. Percentage points for all other integral values of f between 1 and 100 were computed by interpolation. This table overlaps considerably with one by C. M. Thompson [Biometrika **32**, 187–191 (1941); these Rev. **3**, 175]. *H. Chernoff* (Urbana, Ill.).

Carpenter, Osmer. Note on the extension of Craig's theorem to non-central variates. Ann. Math. Statistics **21**, 455–457 (1950).

Let $X_1, \cdots, X_n$ be mutually independent normal variates having a common variance. The author gives necessary and sufficient conditions that a quadratic form in the X's have a noncentral χ^2-distribution and that two quadratic forms in the X's be independent. These results are extended to the case where $(X_1, \cdots, X_n)$ has a nonsingular n-variate normal distribution. [In (1), p. 455, replace $(r-2j)$ by $(r+2j)$.] *D. F. Votaw, Jr.* (New Haven, Conn.).

Cochran, W. G. The comparison of percentages in matched samples. Biometrika **37**, 256–266 (1950).

The author gives an extension of the χ^2 test for comparing percentages of successes in independent samples to the case where the samples are matched and thus not independent. He points out that for the case of two samples the extension has been given by others [e.g., McNemar, Psychological Statistics, Wiley, New York, 1949]. Let the observations be arranged in an $r\times c$ array, say (x_{ij}); x_{ij} is the observed value for the ith member of the jth sample ($x_{ij}=0$ or 1; $i=1, \cdots, r$; $j=1, \cdots, c$); $x_{i1}, \cdots, x_{ic}$ are the observations associated with the c members of the ith matched group. Let $u_i=\sum_j x_{ij}$. The null hypothesis is that the u_i 1's in the ith row are distributed at random among the c columns and that these r distributions are mutually independent, where $u_1, \cdots, u_r$ are regarded as fixed. The sample criterion proposed is

$$Q=c(c-1)\sum_j(T_j-\bar{T})^2/[c(\sum_i u_i)-(\sum_i u_i^2)],$$

where $T_j=\sum_i x_{ij}$ and $\bar{T}=\sum_j T_j/c$. Under weak restrictions on the u's it is shown that, for large r, Q has approximately the χ^2 distribution with $c-1$ degrees of freedom. Accuracy of the approximation for small samples is investigated. An F test is found to be about as satisfactory as the χ^2 test. Subdivision of χ^2 into components is discussed. *D. F. Votaw, Jr.* (New Haven, Conn.).

Hartley, H. O. The use of range in analysis of variance. Biometrika **37**, 271–280 (1950).

In the usual two-way classification of randomized blocks, consider (a) the range of treatment means, (b) the ranges, within blocks, of residuals from treatment means, and (c) the mean of these block ranges. An expression is obtained for the correlation of block ranges. An approximate approach (which involves replacing the distribution of (c) by the chi distribution having the same expectation and variance) provides one and five percent points for (a)/(c) as a statistic for testing the significance of treatments. A statistic of the same type is proposed for testing equality of variances. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Crump, S. Lee. The present status of variance component analysis. Biometrics **7**, 1–16 (1951).

Expository lecture.

Kendall, M. G., and Smith, B. Babington. Factor analysis. J. Roy. Statist. Soc. Ser. B. **12**, 60–94 (1950).

This publication consists of two expository papers and discussion by various statisticians and psychologists. Part 1, written by Kendall, is entitled "Factor Analysis as a Statistical Technique." Factor analysis is considered as a type of "component analysis," which in turn is considered as a kind of "analysis of interdependence." The author treats, as methods of component analysis of correlation matrices, the method of principal components (proving certain properties of the principal components), and the centroid method (proving orthogonality of the resulting factor loading vectors). Factor analysis is treated as component analysis performed on a matrix obtained from the correlation matrix by replacing the diagonal elements by communalities. Bartlett's and Thomson's methods of estimating

factor scores are discussed. In part II: "An Evaluation of Factor Analysis from the Point of View of a Psychologist," Smith discusses the need for factor analysis methods, what one can hope to accomplish by use of these methods, the effect of interchanging the roles of individuals and tests in factor analysis, and other questions relevant to the use and interpretation of factor analysis in psychology.

T. W. Anderson (New York, N. Y.).

Johnson, N. L. Estimators of the probability of the zero class in Poisson and certain related populations. Ann. Math. Statistics **22**, 94–101 (1951).

Let $V_1, \cdots, V_n$ be a sample from a discrete distribution (thought to be Poisson) and let $p = \Pr(V_i = 0)$. This paper compares the two estimates of p: $e^{-\frac{1}{n}\Sigma V_i}$ and n_0/n (where n_0 is the number of zeros among the V's). For this comparison the actual distribution of the V's is assumed to belong to a parametric family of which the Poisson distribution is a limiting case and the loss is taken to be the square of the error.

E. L. Lehmann (Berkeley, Calif.).

Dantzig, George B., and Wald, Abraham. On the fundamental lemma of Neyman and Pearson. Ann. Math. Statistics **22**, 87–93 (1951).

Let $f_i(x)$, $i=1, \cdots, (m+1)$, be $(m+1)$ Borel measurable functions on a Euclidean space R such that $\int_R |f_i(x)|dx < \infty$ for all i. Let $c_1, \cdots, c_m$ be m constants and S the totality of all Borel sets B such that $\int_B f_i(x)dx = c_i$, $i=1, \cdots, m$. The authors (1) prove that if S is not empty then there exists a Borel set B_0 in S such that $\int_{B_0} f_{(m+1)}(x)dx$ is maximal for all sets in S, (2) give necessary and sufficient conditions to characterize the maximal sets B_0. Under mild restrictions these conditions coincide with the sufficient conditions given by Neyman and Pearson [Statist. Res. Mem. London. **1**, 1–37 (1936)].

J. Wolfowitz (Ithaca, N. Y.).

Federer, Walter T. Testing proportionality of covariance matrices. Ann. Math. Statistics **22**, 102–106 (1951).

Let $(\sigma_{ij})_x$, $(\sigma_{ij})_y$, $(\sigma_{ij})_z$ be the variance-covariance matrices of mutually independent vectors x, y, z, respectively, having p-variate normal distributions ($p \geqq 2$). Likelihood ratio criteria for the following hypotheses are presented: (1) $(\sigma_{ij})_x = K(\sigma_{ij})_y$; (2) $(\sigma_{ij})_x = K_1(\sigma_{ij})_y = K_2(\sigma_{ij})_z$ (K, K_1, K_2 are constants whose values are unknown). Extension of the results to the case of r ($\geqq 4$) mutually independent vectors is discussed briefly. Unsolved problems regarding estimation of the constants and extension of the criteria to the case of dependent vectors are mentioned.

D. F. Votaw, Jr.

Petrov, A. A. Test of the hypothesis of the normality of distributions in small samples. Doklady Akad. Nauk SSSR (N.S.) **76**, 355–358 (1951). (Russian)

Let F be a distribution function; let $a_1, \cdots, a_N$; $b_1, \cdots, b_N$ be constants; for each $i=1, \cdots, N$ let $x_{i1}, \cdots, x_{in}$ be a sample of n taken from a population with distribution function $F(a_i x + b_i)$. Let $x'_{i1} < \cdots < x'_{in}$ be the ith sample when ordered, and define $\xi_{ik} = (x'_{ik} - x'_{i1})/(x'_{in} - x'_{i1})$. The author finds the distribution of ξ_{ik}. This distribution depends on F but not on a_i and b_i. If N samples of n are given, and if H is the hypothesis that they are obtained as above, with F prescribed but with a_i and b_i unknown, the author proposes as a test of H a comparison of the calculated and empirical distribution of $\xi_{1k}, \cdots, \xi_{Nk}$ for each k. [The question of mutual dependence of these tests is not discussed.] A different test of H has been discussed by Arley and Buch [Introduction to the Theory of Probability and Statistics, Wiley, New York 1950; these Rev. **11**, 187].

J. L. Doob (Urbana, Ill.).

Hemelrijk, J. A symmetry test. Math. Centrum Amsterdam. Rapport **ZW-1950-015**, 9 pp. (1950). (Dutch)

This brief report considers one of the tests of a group completely described in papers by the author [Nederl. Akad. Wetensch., Proc. **53**, 945–955 = Indagationes Math. **12**, 340–350 (1950); these Rev. **12**, 37]. The author gives a paramrterfree test for the hypothesis H_0, that n values $Z_1, \cdots, Z_n$, be observations of n independent random variables $z_1, \cdots, z_n$, all of which have the same continuous probability distribution which is symmetric with respect to 0. Two theorems are proved. Let u designate the number of positive z which are less than, and v those greater than the median of $|z|$. Theorem 1 states that under hypothesis H_0, the u and v are independently distributed as follows, $P(u=U, v=V|H_0) = 2^{-n}\binom{r}{U}\binom{n-r}{V}$. Theorem 2 establishes conditions that the tests for H_0 be asymptotically decisive with increasing n.

A. A. Bennett (Providence, R. I.).

Blyth, Colin R. On minimax statistical decision procedures and their admissibility. Ann. Math. Statistics **22**, 22–42 (1951).

By means of examining the asymptotic behavior of a sequence of Bayes risks, the usual fixed-sample-size estimates for the means of normal and rectangular populations of known variance are shown to be admissible (as well as minimax) with respect to the class of all sequential estimates having continuous risk functions. Here the risk may be the expectation of a linear combination of the number N of observations and an arbitrary nondecreasing function W of the absolute error of estimation, or it may be either $E(N)$ or $E(W)$ with the other bounded. Some relations are given between these alternative formulations of the problem. An example shows that fixed-sample-size estimates need not be minimax if N is replaced by a nonconvex function of N when defining risk.

J. L. Hodges, Jr. (Berkeley, Calif.).

Koopmans, T. C., and Reiersøl, O. The identification of structural characteristics. Ann. Math. Statistics **21**, 165–181 (1950).

This paper is concerned with the problem of drawing inferences from the hypothetically exact probability distribution of observed variables to the theoretical structure which generates the distribution. Denote by a vector u the latent variables (essentially nonobservable variables like "true" variables, disturbances, etc.) and by a vector y the observed variables. A specification is concerned with the mathematical form (parametric or nonparametric) of the distribution of the latent variables and the structural relationships assumed between u and y. By a structure $S=(F, \phi)$ is meant a probability distribution (pr.d.) $F(u)$ and a particular structural relationship $\phi(y, u)=0$, which permits the unique determination of y almost everywhere from the values of u. The pr.d. $H(y|S)$ of observed variables is uniquely determined by S, and is said to be generated by S. A model is defined as a set of structures S. The specification problem then involves the definition of a model $\mathfrak{S}$ which by hypothesis contains the structure S generating the pr.d. $H(y|S)$.

A fundamental question is now posed: Can a pr.d. $H(y|S)$, generated by a given $S\epsilon\mathfrak{S}$, be generated by one and only one $S\epsilon\mathfrak{S}$? Whether it is true depends always on $\mathfrak{S}$ and fre-

quently on S. If true, it is said that $\mathfrak{S}$ identifies the given structure S. However, if S is not identifiable by $\mathfrak{S}$, some of its characteristics may be. Consider a structural parameter $\theta(S)$ which is a functional of S. Two structures S and S^* are observationally equivalent ($S\sim S^*$) if they generate $H(y|S)=H(y|S^*)$ for all y. Then $\mathfrak{S}$ identifies $\theta(S)$ in S_0, if that parameter has the same value in all structures S_0^*, $S_0\epsilon\mathfrak{S}$ and $S_0^*\sim S_0$. Similar remarks are applicable to characteristics $\chi(S)$ of S other than parameters. It is now shown that identifiability is in principle subject to statistical test. Each structural characteristic $\chi(S)$ partitions $\mathfrak{S}$ into two disjunct subsets $\mathfrak{S}=\mathfrak{S}_\chi+\mathfrak{S}_{\bar\chi}$, such that $\chi(S)$ is uniquely identifiable in S_0 by $\mathfrak{S}$ if $S_0\epsilon\mathfrak{S}_\chi$ and not uniquely identifiable if $S_0\epsilon\mathfrak{S}_{\bar\chi}$. This partition has the property that if $S_0\epsilon\mathfrak{S}_\chi$, then all $S_0^*\sim S_0\epsilon\mathfrak{S}_\chi$. Hence identifiability of $\chi(S)$ in S_0 depends only on the distribution $H(y)=H(y|S_0)$. To the partition of $\mathfrak{S}$ there corresponds a division $\mathfrak{H}=\mathfrak{H}_\chi+\mathfrak{H}_{\bar\chi}$ of the set $\mathfrak{H}=\mathfrak{H}(\mathfrak{S})$ of all pr.d. $H(y|s)$ generated by $S\epsilon\mathfrak{S}$. Then $\mathfrak{H}_\chi$ contains all pr.d. generated by S in which $\chi(S)$ is uniquely identifiable, and $\mathfrak{H}_{\bar\chi}$ contains all pr.d. generated by S for which this property does not hold. Whenever the identification of $\chi(S)$ cannot be decided in the same sense for all $S\epsilon\mathfrak{S}$ as a consequence of either $\mathfrak{S}_\chi=0$ or $\mathfrak{S}_{\bar\chi}=0$, the identifiability of $\chi(S)$ is a property of pr.d. $H(y|S)$. This is equivalent to the hypothesis $H(y|S)\epsilon\mathfrak{H}_\chi$, which in principle is subject to a statistical test under the maintained hypothesis $H(y|S)\epsilon\mathfrak{H}$. The case may arise where the model is defined by general specification supplemented with a number of particular specifications which may be used in combinations to construct alternative models. The general specification is defined as a set $\mathfrak{S}$ which by assumption contains the model $\mathfrak{S}'$ as a subset. Particular specifications can be defined as subsets $\mathfrak{S}_1, \mathfrak{S}_2, \cdots$ of $\mathfrak{S}$ such that the model $\mathfrak{S}'$ is the intersection $\mathfrak{S}'=\mathfrak{S}\cap\mathfrak{S}_1\cap\mathfrak{S}_2\cap\cdots$. Consider a model $\mathfrak{S}$ narrowed to the alternative $\mathfrak{S}'=\mathfrak{S}\cap\mathfrak{S}_1$ by a particular specification $\mathfrak{S}_1$. This specification is called observationally restrictive if $\mathfrak{H}(\mathfrak{S}')$ of all pr.d. $H(y|S')$ generated by $S'\epsilon\mathfrak{S}'$ is a proper subset of $\mathfrak{H}(\mathfrak{S}')$ of all pr.d. $H(y|S)$, $S\epsilon\mathfrak{S}_1$. A statistical test of $\mathfrak{S}_1$ can be constructed by choosing as the hypothesis subject to test $H(y)\epsilon\mathfrak{H}(\mathfrak{S}')$ under the maintained hypothesis $H(y)\epsilon\mathfrak{H}(\mathfrak{S})$. However, $\mathfrak{S}_1$ remains subject to test provided that such particular specifications as are necessary to make $\mathfrak{H}(\mathfrak{S}')$ a proper subset are retained in the model. There follow applications to examples drawn from econometrics and factor analysis. A comparative discussion emphasizes the strong formal similarity of the identification problems to which these examples give rise.

M. P. Stoltz (Providence, R. I.).

Kullback, S., and Leibler, R. A. On information and sufficiency. Ann. Math. Statistics **22**, 79–86 (1951).

The "information" of μ_1 relative to μ_2, where μ_1 and μ_2 are probability measures absolutely continuous with respect to each other, is defined to be $I(1:2)=\int d\mu_1 \log (d\mu_1/d\mu_2)$, this being a natural extension of a concept introduced by Shannon and Wiener in connection with communication engineering. General properties of $I(1:2)$ are explored, and its connection with R. A. Fisher's notion of information in connection with estimation is exhibited. It is demonstrated that all statistics for μ_1 and μ_2, except sufficient statistics, lose "information." A metric, $J(1, 2)=I(1:2)+I(2:1)$, introduced by H. Jeffreys is also discussed. Two suggestive applications of the concepts are mentioned but without critical justification.

L. J. Savage (Chicago, Ill.).

Schutzenberger, Marcel Paul. Sur les rapports entre la quantité d'information au sens de Fisher et au sens de Wiener. C. R. Acad. Sci. Paris **232**, 925–927 (1951).

Let $H(x)$ be a measure of the quantity of information in an observation determining whether the state of a system is or is not in a set of probability x. It is not made quite clear what the exact mathematical properties of H are, but it is stated that under very broad assumptions H has the form $H=xD\log x+(1-x)D\log(1-x)$, where D is an "arbitrary linear operator." If the basic probability distribution depends on a parameter θ, and if $D=\partial^2/\partial\theta^2$, one obtains the Fisher definition of amount of information, used in the problem of estimating θ. If $D=-\log_2$, one obtains the information quantity used by Wiener and Shannon. Other possible versions of D are discussed.

J. L. Doob.

Mathematical Biology

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. VI. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 683–687 (1950).

[For part V see the same vol., 23–25 (1950); these Rev. **12**, 39.] The life of the spermatozoid, and that of the ovule, each is supposed described by a set of m functions, y'^{α} and y''^{α}, respectively, which satisfy a system of linear differential equations of arbitrary order; thence a suitable assumption as to interdependence assures that the $2m$ functions extremize a definite integral. Finally, supposing that $y'^{\alpha}=y''^{\alpha}$ following fertilization, a nonnegative second variation implies stability of the species.

A. S. Householder.

Marchand, Henri. Sur une loi d'union sélective dépendant de l'écart entre la valeur d'un caractère et une valeur optimum. C. R. Acad. Sci. Paris **231**, 1029–1031 (1950).

Marchand, Henri. Influence du degré de dominance du caractère primaire sur l'évolution d'une population soumise à une loi d'union sélective particulière. C. R. Acad. Sci. Paris **231**, 1210–1212 (1950).

These two notes explore the biological evolution of an idealized large bisexual population. One pair of alleles with incomplete dominance control the expected value of a quantitative character S. Roughly speaking, each organism of the population suffers a small selective disadvantage proportional to the square of the departure of its value S from some optimum value. Change in composition of the population from generation to generation, the equilibrium composition appropriate to each set of parameter values, and the rate of approach to equilibrium are worked out. It is emphasized that the behavior of the population in these respects is not monotonic in the degree of dominance.

L. J. Savage (Chicago, Ill.).

Rashevsky, N. Some bio-sociological aspects of the mathematical theory of communication. Bull. Math. Biophys. **12**, 359–378 (1950).

Channel capacity is calculated for cases of interconnected neural chains, and of "social chains." Relations between degree of coordination of a muscular reaction and speed of the reaction, and between amount of knowledge and amount of information, are discussed.

A. S. Householder.

Rapoport, Anatol. Contribution to the probabilistic theory of neural nets. III. Specific inhibition. Bull. Math. Biophys. **12**, 317–325 (1950).

[For part II see the same vol., 187–197 (1950); these Rev. **12**, 431.] The author considers a neuron N_3, stimulated by N_1 and inhibited by N_2, with finite refractory period δ and finite period σ of latent inhibition, where N_1 and N_2 are fired by random stimuli. If xe^{-xt} is the probability that N_1 (or N_2) will fire within time t, given that firing of that neuron had not occurred for at least a time δ, the author solves an integral equation to obtain the probability $P(t)$ that the neuron will fire within time t, nothing being known of the previous history. He now asks whether the frequency x_3 of firing of N_3 has a maximum with respect to x, the same function $P(t)$ being applied to both N_1 and N_3, and finds that there is such provided $\sigma > \frac{1}{2}\delta$. *A. S. Householder.*

Rapoport, Anatol. Contribution to the probabilistic theory of neural nets. IV. Various models for inhibition. Bull. Math. Biophys. **12**, 327–337 (1950).

Dropping the assumption of specifically inhibitory neurons or endings, the author considers the possibility that when a neuron N_3 receives endings from N_1 and N_2, (a) stimulation by either renders it insensitive to the other for a finite time δ (the refractory period), or that (b) firing of both N_1 and N_2 within an interval σ leaves N_3 inactive while firing of either alone causes firing of N_3. Using the same statistical distribution of activity as in the paper reviewed above, the author obtains the average frequency x_3 of firing of N_3 in each of the two cases, exactly in the case (a) and approximately in case (b). *A. S. Householder.*

Mathematical Economics

***Arrow, Kenneth J. Social Choice and Individual Values.** Cowles Commission Monograph No. 12. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. xi+99 pp. $2.50.

This book is concerned with the following problem. A collection of individuals and a set of social alternatives are given and it is assumed that each individual ranks the alternatives in accordance with his preference. Problem: to obtain a "social ordering" of the alternatives as a function of the individual orderings, which will represent the preferences of the community as a whole and which will satisfy certain requirements of compatibility with the preferences of the individuals. The problem is formalized as follows. A set S of alternatives is given. A weak ordering on S is defined to be a relation R which is transitive and such that any two alternatives are comparable. A set of n weak orderings $R_1, \cdots, R_n$ is given corresponding to n individuals. The problem is then to find a function which attaches to each such n-tuple of orderings an ordering R. Such a function is called a "social welfare function," and the ordering R is called the "social ordering." The author now gives a number of requirements which the social welfare function must satisfy, which we paraphrase roughly. (1) If two different sets of individual orderings $R_1, \cdots, R_n$ and $R_1', \cdots, R_n'$ are identical except that a particular alternative x is raised in preference by some of the individuals in the second set of orderings R_i', then this alternative will not be lowered in the corresponding social ordering R'. (2) The relative positions of two alternatives x and y in the social ordering R shall depend only on their relative positions in the individual orderings $R_1, \cdots, R_n$ and not on the positions of alternatives distinct from x and y. A social welfare function is called "imposed" if for some pair of alternatives x and y, xRy for every social ordering R. A social welfare function is called "dictatorial," if there exists an integer i, $1 \leqq i \leqq n$, such that for any $R_1, \cdots, R_n$, the social ordering R is the same as R_i.

The main result of the book can now be stated (General Possibility Theorem): If S contains more than two alternatives then any social welfare function satisfying the first and second conditions must be either imposed or dictatorial. A similar theorem is also proved for cases where restrictions are placed on the allowable individual orderings $R_1, \cdots, R_n$. A large portion of the book is taken up with giving economic justifications for the various axioms and conditions used in setting up the problem. However, the argument for (2) is not convincing. The following simple example may illustrate the difficulty. Two individuals are ranking 100 alternatives. Suppose x and y are two alternatives and suppose the first individual ranks x first and y last, the second ranks y first and x second. It then seems reasonable that the social ordering should rank x above y. On the other hand if the first individual ranks x first and y second, while the second ranks y first and x last the same reasoning would rank y above x in the social ordering. However, the author's second condition requires that x must also be ranked above y in this second case, which seems to contradict common sense. Thus if one accepts the author's remark that the result of the main theorem is "paradoxical" it would seem that paradoxes are already evident in his basic assumptions.

D. Gale (Providence, R. I.).

***Jensen, Arne. Moe's Principle. An Econometric Investigation Intended as an Aid in Dimensioning and Managing Telephone Plant. Theory and Tables.** The Copenhagen Telephone Company, Copenhagen, 1950. 159 pp. (6 plates). 15.00 kroner.

The author is concerned with the problem of determining the optimum number of connecting devices relative to the expense involved and the demand for service in the area. The economic principles on which the investigation is based involve the well-known extremum solution and are taken from Samuelson [Foundations of Economic Analysis, Harvard University Press, Cambridge, Mass., 1947; these Rev. **10**, 555]. In applying these results to the telephone problem the following hypothetical plant is defined. The service area of the plan is subdivided into m areas, each with its own exchange. Hence the plant produces multiple products x_{ij} for several markets. Given a system of tariffs, there will correspond a demand for traffic from exchange i to j equal to $A_{ij}(t)$ erlang. (Traffic, measured in erlang: $ys = A$, where y is the number of calls originated during unit time, and s is the average duration of call). The plant is composed of a number of factors of production (connecting devices in the extended sense). The different stages in the exchanges (factors) are denoted by $1, 2, \cdots, k$ as of outgoing and two-way traffic, and $k+1, k+2, \cdots, 2k$ as of incoming traffic. The number of trunks between exchanges i and j is denoted by $n_{0;ij}$, and the number of connecting devices in stage r, exchange i, is denoted by $n_{r;i}$. The plant hires the requisite installations and staff, and the rent is assumed to be linearly dependent on the number of units hired. Denote by $A_{0;ij}$ and $A_{r;i}$ $(r = 1, 2, \cdots, 2k)$ the amounts of traffic carried by the network and the individual stages of opera-

tion of the exchange. Subscribers' calls are assumed to be distributed at random according to Poisson's law, and the individual connecting devices are assumed to be occupied during periods of time which are distributed exponentially as $P(t)dt = s^{-1}e^{-t/s}dt$. If it is supposed that lost traffic will not originate new calls, and that waiting subscribers will continue to wait, it is shown that the probability of loss (call not completed) for loss systems will be

$$\text{(I)}\qquad E_{1,n}(A) = \frac{A^n/n!}{1+\dfrac{A}{1!}+\cdots+\dfrac{A^n}{n!}},$$

and the average waiting time per call for waiting-time systems will be

$$\text{(II)}\qquad M_n(A) = \frac{s}{n-A}\cdot\frac{\dfrac{A^n}{n!}\cdot\dfrac{n}{n-A}}{1+\dfrac{A}{1!}+\cdots+\dfrac{A^{n-1}}{(n-1)!}+\dfrac{A^n}{n!}\cdot\dfrac{n}{n-A}} = \frac{s}{n-A}E_{2,n}(A),$$

where n is the total number of connecting devices.

For generality introduce the operators $\delta_{0;ij}$ and $\delta_{r;i}$ which assume value zero in waiting-time systems and the value one in loss systems. Similarly, define $\delta'_{0;ij}$ and $\delta'_{r;i}$, which assume value 1 when there exists a trunk route from i to j, or a group of rth stage connecting devices in exchange i, and otherwise zero. The traffic handled on route (i, j) during unit time is

$$x_{ij} = \varphi_{ij} = A_{ij}(1-\delta'_{0;ij}\delta_{0;ij}E_{1,n_{0;ij}}(A_{0;ij})) \times (1-\delta'_{1;i}\delta_{1;i}E_{1,n_{1;i}}(A_{1;i}))\cdots(1-\delta'_{2k;j}\delta_{2k;j}E_{1,n_{2k;j}}(A_{2k;j})).$$

For small values of the loss E_1, a good approximation is

$$\varphi_{ij} = A_{ij}(1-\delta\delta' E_{1,n_{0;ij}}(A_{0;ij}) - \delta'\delta E_{1,n_{1;i}}(A_{1,i}) - \cdots - \delta\delta' E_{1,n_{2k;j}}(A_{2k;j})).$$

The total rental costs, C, will be,

$$C = A + \sum_{i=1}^{m}\sum_{j=1}^{m}\delta' w_{0;ij}n_{0;ij} + \sum_{r=1}^{2k} w_r \sum_{r=1}^{m}\delta' n_{r;i},$$

where $w_{0;ij}$ is the rent for one additional circuit, and ω_r the rent for an additional connecting device in the rth stage. The cost function is determined by selecting that combination of numbers of connecting devices which minimizes total cost for given amounts of traffic. The procedure is standard and requires the minimizing of $G = C - \sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_{ij}(\varphi_{ij} - x_{ij})$, subject to the condition $\varphi_{ij} = x_{ij}$, $i, j = 1, 2, \cdots, m$, where the λ_{ij} are Lagrangean multipliers.

The engineering problem of determining the numbers of connecting devices, $n_{0;\nu\mu}$ and $n_{r;\mu}$, leading to given amounts of traffic can be solved by the usual methods for finding extrema in discrete space. In the present application the work is simplified numerically by employing expressions due to K. Moe:

$$\text{(III)}\qquad F_{1,n}(A) = A(E_{1,n}(A) - E_{1,n+1}(A)) = -A\Delta_n E_{1,n}(A) = -A\nabla_n E_{1,n+1}(A),$$

$$\text{(IV)}\qquad F_{M,n}(A) = y(M_n(A) - M_{n+1}(A)) = -y\Delta_n M_n(A) = -y\nabla_n M_{n+1}(A),$$

where, in general, ∇ and Δ are divided differences with the meaning

$$\nabla f(x_s) = \frac{f(x_{s-1}) - f(x_s)}{x_{s-1} - x_s};\quad \Delta f(x_s) = \frac{f(x_s) - f(x_{s+1})}{x_s - x_{s+1}}.$$

The results may be summarized: (1) For a loss system, the number n of connecting devices in a stage which, carrying a traffic of A erlang, are involved in the handling of a total weighted traffic A_1, with weights λ_{ij}, is determined by the condition: $F_{1,n-1}(A) > wA/A_1 \geqq F_{1,n}(A)$, where w is the rent for one additional unit of the connecting device in question. (2) For a waiting-time system, the number n of connecting devices in a stage which, carrying a traffic of A erlang resulting from y calls in unit time, are involved in a total weighted traffic A_2 is determined by the condition:

$$F_{M,n-1}(A) > wy/(-D_M A_2) \geqq F_{M,n}(A),$$

where $\Delta_M A = D_M A = \nabla_M A$ approximately. Hence, under the given assumptions, the optimal number of connecting devices will be determined by $F_{1,n}(A)$ and $F_{M,n}(A)$ regardless of the revenue function for the plant and regardless of the economic objectives. For a complete determination of the number of connecting devices, the λ_{ij} and wA/A_1 and $wy/(-D_M A_2)$ must be determined. These quantities are dependent on the economic aims of the service, the price policy, and the demand curves for the service. Examples are given of the determination of these constants under different economic aims. There follows a short discussion of seasonal variations in traffic intensity. Useful and comprehensive tables of the functions $E_{1,n}(A)$, $M_n(A)$, $E_{2,n}(A)$, $F_{1,n}(A)$, $F_{M,n}(A)$, and $F_{2,n}(A)$, as defined in (I), (II), (III), and (IV), are provided for selected values of the arguments. The procedure and tables presented are not restricted in application to the telephone industry. *M. P. Stoltz.*

Simpson, Paul B. Risk allowances for price expectations. Econometrica 18, 253–259 (1950).

In discussions of entrepreneurial planning some economists have employed the device of expected prices equal to most probable prices plus or minus risk allowances. In the present paper these adjusted price expectations are interpreted in terms of the theory of subjective risk. The following assumptions are made: (1) The estimated future prices of all commodities affecting the entrepreneur are random variables with means $Ep_i = \bar{p}_i$, and variances and covariances $E(p_i - \bar{p}_i)(p_j - \bar{p}_j) = \sigma_{ij}$. (2) The planned quantities are denoted by $+q_i$ for sales and $-q_i$ for purchases. The feasible sets (q_i) are given by a production function (I) $Q(q_1, q_2, \cdots, q_n) = 0$. (3) The calculated profit (II) $V = \sum p_i q_i$ is taken as a normally distributed random variable with mean $\bar{V} = \sum q_i \bar{p}_i$ and variance $S^2 = \sum_{i,j} q_i q_j \sigma_{ij}$. (4) There exists an analytic function $\phi(\bar{V}, S)$ for each entrepreneur, which orders his preferences for assigned values of $\bar{V}$ and S. (5) Each p_i is independent of q_j. The function ϕ may be maximized subject to the constraint (I) by usual methods. Only necessary conditions are given here which precludes certain solutions. The resulting equations along with (I) and (II) yield $n+2$ equations to solve for $n+2$ unknowns. If the maximizing solution is expressed in terms of the slope of the indifference curve at equilibrium, we have

$$\text{(III)}\qquad \frac{\partial Q/\partial q_i}{\partial Q/\partial q_j} = \frac{\bar{p}_i - (1/S)(d\bar{V}/dS)(\sum_j q_j \sigma_{ij})}{\bar{p}_j - (1/S)(d\bar{V}/dS)(\sum_i q_i \sigma_{ij})},$$

which expresses the fact that equilibrium of the firm is given by the equality of the ratio of the marginal rates of substitution for two inputs or outputs and the ratio of expected prices plus or minus risk allowances expressed as certainty equivalents. Conditions for conservative behavior are specified, and a further interpretation in terms of utility is given. The effect of limited capital resources on behavior is discussed. *M. P. Stoltz* (Providence, R. I.).

TOPOLOGY

Quast, J., and Schuh, Fred. A number of path problems. Simon Stevin **27**, 201–211 (1950). (Dutch)

The authors consider a graph which has $n+1$ nodes $C_0, C_1, \cdots, C_n$ and mn branches: m branches joining C_{i-1} and C_i for each i from 1 to n. They seek the number of paths from C_0 to C_n, never going twice along the same branch but possibly going several times through the same node; e.g., for $n=1$ it is $m+m(m-1)(m-2)+m(m-1)(m-2)(m-3)(m-4)+\cdots$. They obtain the number as a function of n for $m=3, 4, 5$; e.g. for $m=4$ it is $\{(1+c)(26+2c)^n+(-1+c)(26-2c)^n\}/2c$, where $c=\sqrt{145}$. They also consider five other families of graphs. *H. S. M. Coxeter* (Toronto, Ont.).

Gallai, T. On factorisation of graphs. Acta Math. Acad. Sci. Hungar. **1**, 133–153 (1950). (English. Russian summary)

A subgraph of a given graph G is a graph G' whose edges and vertices are edges and vertices respectively of G, with the same incidence relations as in G. A subgraph G' of G is a factor of G of degree n if each vertex of G has degree n, that is, is incident with just n edges, in G'. The author presents a theory of factors using the method of alternating paths introduced by Petersen [Acta Math. **15**, 193–220 (1891)]. This enables him to give a unified treatment of the theorems contributed by Petersen, König, Baebler, Hall, Radó, and the reviewer. Thus he derives the necessary and sufficient condition for the existence of a factor of degree 1 in a general graph, and the simpler condition which applies to even graphs, in a new way. He states that he has derived a necessary and sufficient condition for the existence of a factor of degree 2 in a general graph which he will discuss in a later paper. A graph is regular if the degrees of its vertices are all equal. The author extends the theory of the factors of regular graphs by giving a necessary and sufficient condition for the existence of a factor of any given degree in a regular graph. *W. T. Tutte* (Toronto, Ont.).

Kaluza, Theodor, jr. Zur Rolle der Epsilonzahlen bei der Polynomdarstellung von Ordinalzahlen. Math. Ann. **122**, 321–322 (1950).

This is a further example of the application to set-theoretical studies of the author's results on infinite graphs [Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 3; Math. Ann. **122**, 235–258 (1950); these Rev. **11**, 196; **12**, 434]. *W. T. Tutte.* (Toronto, Ont.).

Kelley, J. L. The Tychonoff product theorem implies the axiom of choice. Fund. Math. **37**, 75–76 (1950).

Let (T) denote the theorem of Tychonoff (that the product of compact spaces is compact) and let (C) denote the axiom of choice in the form that asserts that a product of nonvoid sets is nonvoid. The author shows that (T) implies (C). *A. D. Wallace* (New Orleans, La.).

Umegaki, Hisaharu. On the uniform space. Tôhoku Math. J. (2) **2**, 57–63 (1950).

A uniform space E is said to satisfy the condition (*) if the defining system $\{V_\alpha\}$ of neighborhoods of the diagonal Δ in $E\times E$, contains a sequence $\{V_{\alpha_n}\}(n=1, 2, \cdots)$ such that $\bigcap V_{\alpha_n}=\Delta$. The author extends a result of Dieudonné by proving that if E satisfies (*) then E has a uniform structure (compatible with its topology) in which it is complete. If E satisfies (*), or is fully normal, various compactness properties of E are shown to be equivalent [the hypothesis of full normality could be weakened; cf. Arens and Dugundji, Portugaliae Math. **9**, 141–143 (1950); these Rev. **12**, 434]. A connected uniform space which uniformly locally has a countable base is metrizable. If E satisfies (*) and is uniformly locally countably compact, then E is metrizable [the author asserts, in theorem 2, that E is then uniformly metrizable; this is false, for example, when E is the real line and $\{V_\alpha\}$ is the family of all neighborhoods of Δ]. The author answers a question raised by Kakutani by observing that the space of countable ordinals is locally compact but not uniformly locally compact in its unique uniform structure. *A. H. Stone* (Manchester).

Votavová, Libuše. Conditions of compactness for Alexandroff's space αP. Acta Fac. Nat. Univ. Carol., Prague no. **194**, 29–33 (1948). (English. Czech summary)

Let P be a regular space, and let $\mathcal{G}$ be the lattice of open sets. Call a dual ideal A in $\mathcal{G}$ regular if for a in A there is a b in A whose closure is contained in A. The class of maximal regular dual ideals, with a natural topology becomes a Hausdorff extension αP of P [P. Alexandroff, Rec. Math. [Mat. Sbornik] N.S. **5**(47), 403–423 (1939); these Rev. **1**, 318]. The author gives necessary and sufficient conditions for αP to be H-closed and for αP to be compact. The latter is as follows: If $A_1, A_2, \cdots$ is a sequence of sets in P such that A_n is deep within A_{n-1}, then there is a continuous real-valued function f such that $\lim \inf_{x\in A_n} f(x) > \sup_{x\notin A_1} f(x)$. There is presented an example due to Katětov for which αP is not compact. *R. Arens* (Los Angeles, Calif.).

Gerolini, Annamaria. Compactification des espaces séparés. C. R. Acad. Sci. Paris **232**, 1056–1058 (1951).

By considering the class G of filters in R obtained by intersecting with R the filters F_x, where F_x is the class of sets S such that $x\in$ int S, for all x in $B-R$ the author is led to a (necessary) condition which together with regularity implies that a compact B containing R exists. The condition requires the existence of a set G of filters with certain properties which the G mentioned above has. The construction of B does not employ the axiom of choice, and does not necessarily lead to the Stone-Čech compactification. Indeed, when R is locally compact it leads to the "point at infinity" compactification, when G is chosen suitably. *R. Arens* (Los Angeles, Calif.).

Nagata, Jun-iti. On topological completeness. J. Math. Soc. Japan **2**, 44–47 (1950).

A short proof is given of E. Čech's theorem [Ann. of Math. (2) **38**, 823–846 (1937)] that a space is completely metrizable if and only if it is a G_δ in some (bi)compact Hausdorff space. It is based on an observation of N. A. Shanin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 154–156 (1943); these Rev. **5**, 46], concerning when a T_1-space is the intersection of at most $\mathfrak{m}$ open sets in its Wallman compactification. *R. Arens* (Los Angeles, Calif.).

Majstrenko, Petro. On certain local properties of a topological space associated with a pseudo-metric. I. Nederl. Akad. Wetensch., Proc. **53**, 1199–1210 = Indagationes Math. **12**, 432–443 (1950).

The author's results about ~~semipseudo matrices~~ will be reported for brevity for the special case of a bounded

semi-pseudo-metric spaces

metric d. A property P of sets is spherically hereditary if a sphere has it whenever some including sphere has it. An element E_{ar} about a point a is the largest sphere about a having P, supposing there are such spheres, and its radius r is the radius of validity of P at a (this theory is intended to generalize that of analytic function elements). An ordered pair E_{ar}, E_{bs} of elements is a t-direct continuation ($0<t\leqq 1$) if $d(a, b)<tr$ and ${}_tE_{ar}$ is the class of elements which are (form) t-direct continuations of (with) E_{ar}. Iteration n times yields $({}_tE_{ar})^n$. A result will now be stated. Suppose for every set of m elements in $U=({}_tE_{ar})^n$ there is at least one pair of reciprocally t-direct continuations. Then there exists a set B of less than m elements of U such that (a) no two are t-direct continuations, and (b) every element of U is a reciprocally t-direct continuation of some element in B. A collection B of this type is called a basis for U, and m is called an $n-r$ integer. Another result: If all $n-r$ integers exist then $({}_tE_{ar})^\infty$ has a countable basis. It is finally proved that all $n-r$ integers exist if a $2-r$ integer exists.

R. Arens (Los Angeles, Calif.).

Katětov, M. On mappings of countable spaces. Colloquium Math. 2, 30–33 (1949).

It is proved that a countable regular space admits a one-to-one continuous mapping onto a compact space if and only if it contains no dense-in-itself subset. [In connection with this problem, see also the reviewer's note in Bull. Amer. Math. Soc. 55, 421–426 (1949); these Rev. 10, 616.]

E. Hewitt (Uppsala).

Villegas Mañé, Cesáreo. A theorem on the local inversion of transformations. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 2, 1–41 (1950). (Spanish. English summary)

The author extends known theorems on local inversion and prolongation of interior transformations due primarily to Stoïlow. Results are established from which it follows that if $x=x(t)$ is a mapping from a regular locally compact weakly separable topological space E to Euclidean E_n which is locally topological in a deleted neighborhood of $\alpha\epsilon E$, there exists an (arbitrarily small) open sphere K with center $a=x(\alpha)$ and a region $D\subset U$ containing α such that $D-(\alpha)$ is the union of m disjoint domains D_h with $D_h+(\alpha)$ mapping topologically onto K under $x(t)$ for $n\geqq 3$. For $n=2$ the mappings from the $D_h+(\alpha)$ to K are topologically equivalent to power transformations as in the theorems of Stoïlow. Applications are made to the uniformization of functions in the situations of classical analysis. *G. T. Whyburn.*

Morita, Kiiti. On the dimension of normal spaces. II. J. Math. Soc. Japan 2, 16–33 (1950).

After some preparatory theorems the author proves the following. Let R be normal with Lebesque dimension at most n, and let $\{G_a; a\epsilon A\}$ be a locally (i.e. neighborhood) finite covering. Then each G_a can be so diminished as to yield an open covering of order at most $n+1$. This result was announced by Dowker [Bull. Amer. Math. Soc. 52, 243 (1946)] as the author observes. He also refers to a paper of Dowker [Amer. J. Math. 69, 200–242 (1947); these Rev. 8, 594] of which he has heard but has not seen. A related theorem is this. Let $\{F_a\}$ be a closed covering of a normal space R where dim $F\leqq n$, and suppose each $F_a\subset G_a$ where G_a is open and such that a subsystem of $\{G_a\}$ having a smaller power is locally finite. Then dim $R\leqq n$. This in turn implies the "sum-theorem" (for Lebesgue dimension). The main theorem of the paper is this. Let $\{F_a\}$ be closed, $F_a\subset G_a$ where the G_a form a locally finite system of open sets. Let $A\subset R$ have dim $A\leqq n$, A being closed. Then there exist two systems of open sets U_a, V_a such that $F_a\subset V_a$, $V_a^-\subset U_a\subset G$ and the order of the system $A(U_a^- - V_a)$ is not greater than n. An application to metric spaces M shows that dim $M\leqq n$ implies ind $M\leqq n$ where ind refers to the inductive Menger-Urysohn dimension. When every open covering of M has a star-finite refinement one also has that dim $M\leqq n$ only if ind $M\leqq n$. The theorems on mappings into spheres found by Hurewicz and Wallman [Dimension theory, Princeton University Press, 1941, p. 84, corollary, p. 87, (B), and p. 88, (C); these Rev. 3, 312] are generalized to normal spaces, using Lebesgue dimension and the Alexandroff-Hopf definition of homotopy. There are one or two places where the arguments are so concise that it would be helpful to have at hand a forthcoming paper of the author to which references are made. The present paper generalizes, presumably from finite coverings, this other paper. *R. Arens.*

Tola Pasquel, José, and Abuauad, César. On the equivalence of three definitions of continuity of functions in spaces in which convergent sequences may admit more than one limit. Revista Ci., Lima 51, nos. 1–2, 21–28 (1949). (Spanish)

The spaces in question are spaces L of Fréchet except that uniqueness of limit is not required. The definitions of continuity are (1) if $x\epsilon A^-$ then $f(x)\epsilon f(A)^-$, (2) if x_n converges to x then $f(x_{n_i})$ converges to $f(x)$ for some subsequence, and (3) if x_n converges to x then $f(x_n)$ converges to $f(x)$. A simple condition on the range space is necessary and sufficient that (1) and (2) be equivalent, and another does the same for (2) and (3). *R. Arens* (Los Angeles, Calif.).

Swingle, Paul M. The closure of types of connected sets. Proc. Amer. Math. Soc. 2, 178–185 (1951).

Given a connected plane domain D, the author presents examples of various types of connected sets which are dense in D. One such result is the following: If the hypothesis of the continuum is true, then D contains a widely connected subset B which is also a biconnected set without dispersion point, and such that $\bar{B}=\bar{D}$. Various examples of a similar nature are obtained without use of the hypothesis of the continuum. Additional results concern the effect of adding single points to various types of connected sets.

E. E. Floyd (Charlottesville, Va.).

Hamilton, O. H. A fixed point theorem for pseudo-arcs and certain other metric continua. Proc. Amer. Math. Soc. 2, 173–174 (1951).

A chain Y is a collection $y_1, y_2, \cdots, y_k$ of disjoint open sets (in a compact metric space) such that y_i and y_j have a boundary point in common if and only if i and j are identical or consecutive integers. Then $C(Y)$ is the closure of $\bigcup y_i$, and ΔY is the maximum diameter of the y_i's. The author shows that if $Y_1, Y_2, \cdots$ is a sequence of chains, such that (1) $C(Y_{i+1})\subset C(Y_i)$ for each i, and (2) $\lim_{i\to\infty}\Delta Y_i=0$, then the continuum $\bigcap C(Y_i)$ has the fixed-point property under continuous transformations. This theorem applies to various indecomposable continua defined by Brouwer [Math. Ann. 68, 422–434 (1910)], Knaster [Fund. Math. 3, 247–286 (1922)], and the reviewer [Trans. Amer. Math. Soc. 63, 581–594 (1948); these Rev. 10, 56]. *E. E. Moise.*

Valentine, F. A. Arcwise convex sets. Proc. Amer. Math. Soc. 2, 159–165 (1951).

The space is the Euclidean plane E_2. An arc C is convex if it lies in the boundary of its convex hull. A set S is arcwise

convex if each pair of points of S can be joined by a convex arc lying in S. A continuum S is unilaterally connected if each pair of points x, y of S lies in a subcontinuum of S lying in a closed half-plane bounded by the line through x and y. If $S\subset E_2$, then $K(S)$ is the set of all points x such that every point of S can be joined to x by a convex arc in S. Theorems: (1) If S is unilaterally connected, then each component of E_2-S is arcwise convex, and each pair of points in the unbounded component K of E_2-S can be joined by a 3-sided polygonal line in K. (2) If S is a simply connected closed set in E_2, then $K(S)$ is arcwise convex.

E. E. Moise (Ann Arbor, Mich.).

Viola, Tullio. Criteri di compattezza per aggregati d'insiemi elementari di uno spazio euclideo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 48–55 = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **282** (1950).

Let C and D be compact subsets of k-dimensional Euclidean space R^k. Let $\alpha(C, D)$ be the Hausdorff distance between the sets C and D [see for example Alexdroff and Hopf, Topologie. I., Springer, Berlin, 1935, p. 112]. Let fC denote the boundary $C'^{-}\cap C$ of C. Let $\beta(C, D)=\max[\alpha(C, D), \alpha(fC, fD)]$. It is easy to see that β is a metric for the space of all compact subsets of R^k. The author shows that a sequence $\{C_n\}_{n=1}^{\infty}$ of such sets converges to a limit in the metric β if and only if the following conditions hold: (1) For every $\epsilon>0$, there exists an integer $N(\epsilon)$ such that for $m, n>N(\epsilon)$, $\alpha(C_m, C_n)<\epsilon$; (2) for every $\epsilon>0$, there exists a positive real number $\rho(\epsilon)$ such that for every positive integer n and every point $p\epsilon fC_n$, there exists an open sphere S of radius ρ contained in the open sphere of radius ϵ and center p such that $S\cap C_n=0$. A second, similar, theorem is proved, along with two theorems giving necessary and sufficient conditions that every subsequence of a sequence should contain a subsubsequence converging to a limit.

E. Hewitt (Uppsala).

Vol'pert, A. I. An elementary proof of Jordan's theorem. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 168–172 (1950).

Filippov, A. F. An elementary proof of Jordan's theorem. Uspehi Matem. Nauk (N.S.) **5**, no. 5(39), 173–176 (1950). (Russian)

Hu, Sze-tsen. Chain transformations in Mayer chain complexes. Compositio Math. **8**, 251–284 (1951).

Let M and N be two Mayer complexes with the same group of operators W. The author considers equivariant chain-mappings of M into N, the basic tool being the obstruction cocycle of Eilenberg and MacLane [Trans. Amer. Math. Soc. **65**, 49–99 (1949); these Rev. **11**, 379]. Let Q be a segregated subcomplex of M; it is assumed throughout that W operates freely on the chain-groups $C_i(M)$, and a W-base of $C_i(M)$ intersected with $C_i(Q)$ gives a W-base for the latter. Theorems on extensions of equivariant chain-maps and chain homotopies lead to the classification theorem: If $C_i(M)=C_i(Q)$ unless $m<i\leqq n$, the equivariant homotopy classes of equivariant chain-maps $M\rightarrow N$ agreeing with a fixed map on Q are in a one-to-one correspondence with the elements of the direct sum $T_{m+1}\oplus\cdots\oplus T_n$, where each T_i is a factor group of two subgroups of the equivariant cohomology group $H_e^i(M-Q, H_i(N))$. Applications to derive chain-map forms of Hopf's and Steenrod's classification theorems, and a slight strengthening of a theorem of Eilenberg and MacLane on homology groups of groups are also given.

J. Dugundji (Los Angeles, Calif.).

Hu, Sze-tsen. The equivalence of fiber bundles. Ann. of Math. (2) **53**, 256–275 (1951).

In this paper the author shows that the problem of whether or not two given fibre bundles are equivalent can be replaced by a problem of whether or not there exists a cross-section of a third bundle associated with the two given bundles. Let $E=\{B, X, p, Y, G\}$ be a fibre bundle with bundle space B, base space X, projection p, fibre Y, and group G, and let $E'=\{B', X, p', Y, G\}$ be another bundle with the same base space, fibre, and group. The author tells how to associate with the bundles E and E' another fibre bundle E^*, called the Ehresmann bundle of E and E'. The Ehresmann bundle $E^*=\{B^*, X, p^*, G, G^*\}$ has the same base space as E and E', and the fibre of E^* is the underlying space of the topological group G. The group G^* of the bundle E^* is the subgroup of the group of all homeomorphisms of G which is generated by the inner automorphisms and the left and right translations. An equivalence between E and E' is a homeomorphism h of B onto B' which maps fibres onto fibres, and such that (speaking roughly) the restriction of h to a single fibre represents an element of the group G. The Ehresmann bundle has the important property that if X_0 is any subspace of the common base space X, then the equivalences between the parts of E and E' which lie over X_0 may be put into one-to-one correspondence in a natural way with the cross-sections of the part of E^* over X_0. Thus the problem of defining an equivalence between E and E' is equivalent to the problem of defining a cross-section of E^*; in case the base space X is triangulable, it is natural to approach this latter problem by using the theory of "obstructions" [see Steenrod, The Topology of Fibre Bundles, Princeton University Press, 1951, part III; these Rev. **12**, 522]. The author shows that the set of all n-dimensional obstruction elements to cross-sectioning the bundle E^* is a coset of a certain subgroup of the cohomology group in which it is contained (provided, of course, it is nonvoid).

W. S. Massey (Providence, R. I.).

Hu, Sze-tsen. Extensions and classification of maps. Osaka Math. J. **2**, 165–209 (1950).

A pair (M, M_0) consisting of a topological space M and a subspace M_0 is said to dominate another pair (X, X_0) if there exist continuous maps ξ: $(X, X_0)\rightarrow(M, M_0)$ and η: $(M, M_0)\rightarrow(X, X_0)$ such that the composite map $\eta\circ\xi$ is homotopic to the identity map $(X, X_0)\rightarrow(X, X_0)$. Let C_0 denote the class of all pairs (X, X_0) each of which is dominated by some pair (M, M_0) consisting of a simplicial polytope M and a closed subpolytope M_0 such that the dimension of $M-M_0$ is finite. The class C_0 contains the following important subclasses: (1) the set of all simplicial pairs (X, X_0) for which dimension $(X-X_0)$ is finite; (2) the set of all pairs (X, X_0) such that both X and X_0 are compact A.N.R.'s (absolute neighborhood retracts). (3) the set of all pairs (X, X_0) such that X and X_0 are finite dimensional A.N.R.'s, and X_0 is a closed subset of X.

The paper under review gives an exposition of the theory of "obstructions" for the following two problems. Let R be an arbitrary arcwise connected topological space and (X, X_0) a pair belonging to the class C_0. The first problem is as follows: Given a map f: $X_0\rightarrow R$, does there exist a continuous extension of f to all of X? The second problem

is as follows: Given two maps f_0, f_1: $X \to R$ such that $f_0|X_0 = f_1|X_0$, what are necessary and sufficient conditions that f_0 be homotopic to f_1 relative to X_0? Singular cohomology theory is used exclusively, and where necessary the author assumes that the fundamental group of R operates trivially on the higher homotopy groups, so as to avoid the need for cohomology theory with local coefficients. One of the main tools used is the geometric realization of the total singular complex of a topological space [cf. J. H. C. Whitehead, Ann. of Math. (2) **52**, 51–110 (1950), pp. 98–106; J. B. Giever, ibid. **51**, 178–191 (1950); these Rev. **12**, 43; **11**, 379]. The theorems obtained are much like those which other authors have obtained on the general theory of obstructions [e.g., Eilenberg, ibid. **41**, 231–251 (1940); Olum, ibid. **52**, 1–50 (1950); Steenrod, The Topology of Fibre Bundles, Princeton University Press, 1951, part III; these Rev. **1**, 222; **12**, 120, 522]. All the results are of a general nature; no attempt is made to give effective methods for the determination of obstructions in any particular class of problems. There is a rather complete bibliography of the literature on this subject at the end of the paper.

W. S. Massey (Providence, R. I.).

Whitehead, J. H. C. Omotopia. Boll. Un. Mat. Ital. (3) **6**, 36–49 (1951).
Expository lecture.

GEOMETRY

Bagchi, Hari Das. On rational (or Heron) triangles. Math. Student **18** (1950), 33–36 (1951).

Toscano, L. Sur un triangle associé à un triangle donné. Mathesis **60**, 9–14 (1951).

Ramakrishnan, V. Orthopolar theory & Feuerbach's theorem. Math. Student **18** (1950), 25–26 (1951).

Inglada, Vicente. Complex investigation of pairs of related triangles. Math. Notae **9**, 133–142 (1949). (Spanish)

With the complex coordinates of two triangles $A_1B_1C_1$, $A_2B_2C_2$ the author forms three invariants which he uses to derive a number of properties of the two triangles. The following may be considered typical. If two triangles are metaparallel (or orthological) in two ways, they are metaparallel (or orthological) in three ways. If the circles $C_2A_1B_1$, $A_2B_1C_1$, $B_2C_1A_1$ have a point in common, the same holds for the circles $C_1A_2B_2$, $A_1B_2C_2$, $B_1C_2A_2$.

N. A. Court.

Thomas, J. M. Geometrical solution of spherical triangles. Amer. Math. Monthly **58**, 151–158 (1951).

Egerváry, E. On the Feuerbach-spheres of an orthocentric simplex. Acta Math. Acad. Sci. Hungar. **1**, 5–16 (1950). (English. Russian summary)

A simplex of n points in a space of $n-1$ dimensions is orthocentric if its altitudes are concurrent. With reference to such a simplex the author uses a system of barycentric coordinates to extend to an orthocentric group of $n+1$ points in an $n-1$ dimensional space some of the properties of such groups in spaces of two and three dimensions, with emphasis on the properties of the nine-point circle and the twelve-point spheres. It is shown, for instance, that the equality of the circumcircles of an orthocentric group of triangles, a property which does not hold for an orthocentric group of tetrahedrons, has its analog in any even dimensional space. [Additional reference: Raynor, Amer. Math. Monthly **41**, 424–438 (1934).]

N. A. Court.

Yates, Robert C. Centre of curvature for the conics. Math. Gaz. **35**, 19–22 (1951).

Fabricius-Bjerre, Fr. The osculating conics of Steiner's hypocycloid. Elemente der Math. **6**, 29–30 (1951).

Mancill, J. D. Plane areas by complex integration. Amer. Math. Monthly **58**, 232–238 (1951).

Tienstra, J. M. The normal section of the ellipsoid. Bull. Géodésique **1951**, 7–21 (1951).

Lietzmann, W. Möglichkeiten und Grenzen einer Veranschaulichung mehrdimensionaler Geometrie. Math.-Phys. Semesterber. **2**, 117–125 (1951).

Hjelmslev, Johannes. An old problem in a new light. Mat. Tidsskr. B. **1950**, 1–5 (1950). (Danish)

Given a convex polygon $A_1 \cdots A_n$ with exterior angles α_i, let S be the centre of gravity of the points A_i when A_i is given the weight $\sin 2\alpha_i$ and let $a(P)$ be the area of the pedal polygon of P with respect to the given polygon. Then the locus of P for constant $a(P)$ is a circle with its centre in S. This elementary proposition of Steiner [Werke, vol. 2, Reimer, Berlin, 1882, p. 112] is generalized in the assumption that the given polygon is an arbitrary ordered set of straight lines, in which two consecutive ones may be parallel.

F. J. Terpstra (Bandung).

Tavora, Elysiario. Topics in theoretical crystallography. Anais Acad. Brasil. Ci. **22**, 35–50 (1950). (Portuguese)

Tavora, Elysiario. Matrices of dyadics. Anais Acad. Brasil. Ci. **22**, 235–243 (1950). (Portuguese. English summary)

Use of dyadics in the application of group theory to crystallography has resulted in elegant solutions of some of the problems [cf., e.g., Zachariasen, Theory of X-ray Diffraction in Crystals, Wiley, New York, 1945]. The purpose of the present papers is to amplify Zachiariasen's preliminary chapter on dyadics. The first paper deals with the application of dyadics to symmetry problems (characteristic dyadics, reciprocal dyadics, versors, scalar and vector products), the application to crystallography being kept well in the foreground. The second paper deals with the deduction of the matrix elements of the dyadic. The chief application is in finding the equivalent points of a given space-group.

H. A. Thurston (Bristol).

Santaló, L. A. Some inequalities between the elements of a tetrahedron in non-Euclidean geometry. Math. Notae **9**, 113–117 (1949). (Spanish)

It was proved by Pólya [Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, v. 2, Springer, Berlin, 1925, p. 166, problem 17], in the case of a tetrahedron situated in Euclidean 3-space, that if a, A denote respectively the length of an arbitrary edge of the given tetrahedron and the corresponding dihedral angle, then (1) $\frac{1}{3}\pi < r < \frac{1}{2}\pi$ where $r = (\sum aA)/(\sum a)$. The author shows that a similar proof establishes (1) in the case of a tetrahedron of volume V situated in non-Euclidean 3-space of constant curva-

ture K, provided that the definition of r is changed to $r=(\sum aA-2KV)/(\sum a)$. *L. C. Young* (Madison, Wis.).

★Borsuk, Karol. Geometria analityczna w n wymiarach. [Analytic Geometry in n Dimensions]. Monografie Matematyczne, Tom XII. Warszawa-Wrocław, 1950. iv+448 pp.

This book is divided into three parts. The first part contains eight chapters and deals with the following topics: points and vectors in Euclidean real space C_n, Cauchy-Schwartz inequality, hyperplanes (matrices) and their mutual position, Cartesian product, complexes, simplexes, simplicial complexes, triangulation, orthogonal transformations, invariance of angle by a similitude, affine groups, homeomorphism, covariant and contravariant vectors, elementary properties of conic sections and quadrics, and some examples of algebraic curves of higher degree. The second part is divided into 6 chapters and deals chiefly with real projective space P_n and real Möbius space (stereographic projection). Within this frame the following topics are introduced and discussed: affine group as a subgroup of a projective group, hyperplanes in P_n, cross ratio, Pappus theorem, complete quadrilateral, projective mapping, perspectivity, projective mapping as homeomorphism, algebraic hypersurfaces with application to conic sections and quadrics (projectively distinct quadrics), duality and its application, conformality of stereographic projection, Möbius space M_n, its spherical mapping on itself, inversion and its elementary properties. The last part deals with the complex space (metric, affine, or projective) whose underlying group of transformation is a real one. Consequently, most of the properties discussed in the first two parts are valid here too. Furthermore, the following topics are dealt with: Laguerre's formula for an angle, isotropic points, and directions. The main purpose of this part is a complete classification of quadratic figures in the metric, affine, or projective case. The book written by a topologist, is not only an introduction to analytic geometry but at the same time an easy introduction to basic topological notions through analytic geometry. It is worthwhile mentioning that even in the preface to the book the author deals with notions of sets, their intersection and union, mapping and its inversion, groups and subgroups (Abelian or not) and so on.

V. Hlavatý (Bloomington, Ind.).

Rozenfel'd, B. A., and Yaglom, I. M. On the geometries of the simplest algebras. Mat. Sbornik N.S. **28**(70), 205–216 (1951). (Russian)

The two-dimensional Euclidean, spherical, and hyperbolic distances may be written in a well-known manner in terms of complex numbers. This idea can be carried over to any algebra A over the field R of the real numbers with the following properties: There exists an involution $\alpha\rightarrow\bar{\alpha}$ of A on itself such that $\overline{\alpha+\beta}=\bar{\alpha}+\bar{\beta}$, $\overline{\alpha\beta}=\bar{\beta}\bar{\alpha}$, $\bar{\bar{\alpha}}=\alpha$. The algebra has a unit. If $\bar{\alpha}=\alpha$, then α is a real number times the unit; $\alpha\bar{\alpha}$ is a nondegenerate quadratic form. Examples are $R(i,j)$, consisting of quaternions, with base 1, i, j, k, where $k=ij$, $i^2=-1$, $j^2=-1$, and $ij=-ji$; the dual numbers $R(e)$ with base 1 and e, where $e^2=1$; and $R(i,e)$ with base 1, i, e, and f, where $f=ie$, $i^2=-1$, $e^2=1$, and $ie=-ei$. It is shown that geometries can be defined in terms of these algebras which are similar to the above mentioned forms of the Euclidean, spherical, and hyperbolic geometries. The geodesics are discussed. The procedure can be generalized to higher dimensions and leads to some new geometries as well as known ones mostly encountered in the theory of Lie groups. The geodesics and completely geodesic manifolds of these geometries are discussed. It is shown how special cases of these geometries lead to well-known spaces like the symplectic space. *H. Busemann* (Los Angeles, Calif.).

Kustaanheimo, Paul. A note on a finite approximation of the Euclidean plane geometry. Soc. Sci. Fenn. Comment. Phys.-Math. **15**, no. 19, 11 pp. (1950).

In this note the author considers the finite field of residues modulo a prime $p\equiv 3 \pmod 4$. Here -1 is a quadratic non-residue and we may define an ordering: $u>v$ if and only if $u-v$ is a quadratic residue. This ordering has the properties: (1) Either $u>v$ or $v>u$, but not both; (2) given any u, then there exist v, w such that $u>v$, $w>u$. This ordering cannot be transitive, but if the first r primes 2, 3, $\cdots$, q_r are quadratic residues of p then for numbers in the range $a, \cdots, a+q_r$ the ordering is transitive. The affine finite plane with coordinates from $GF(p)$ for such a prime p is shown to resemble locally the real Euclidean plane, where by "locally" is meant points whose coordinates lie in the limited domain. It is conjectured that such a finite geometry may be the basis for explaining the discreteness in the phenomena of modern physics. *M. Hall.*

Lesieur, Léonce. Les fondements de la géométrie. Revue Sci. **88**, 114–120 (1950).

This paper sketches lattice-theoretic foundations of finite-dimensional projective and affine geometries. No references are made to the detailed work in this connection due to Birkhoff [Ann. of Math. (2) **36**, 743–748 (1935)] and K. Menger [ibid. **37**, 456–482 (1936)], the latter work being especially pertinent. The author assumes that (1) the elements of the geometries (linear subspaces) form a lattice with first and last elements, (2) if P is a point (an element that covers 0) and $B\subset A$, then $A(B+P)=B+AP$ (a weakened Dedekind postulate which is equivalent to the "axiom of exchange": if P is a point and $P\not\subset B$, then $B+P$ covers B), (3) each element is the union of a finite number of its points. Adjoining the dual of (2) yields a finite projective geometry (the dual of (3) is a theorem). Affine geometry is obtained by adjoining to (1), (2), (3) the assumption that $B\subset A$ implies $A(B+H)=B+AH$ whenever the hyperplane H (dual of a point) is such that $AH\neq 0$. The paper concludes with brief remarks about Desargues and Pascal planes, and properties of the associated number fields. *L. M. Blumenthal* (Columbia, Mo.).

Weitzenböck, R. W. On the line-comitants of a space cubic. Proc. Cambridge Philos. Soc. **47**, 46–48 (1951).

Primrose [same Proc. **46**, 195–198 (1950); these Rev. **11**, 455] recently determined among other things a projective invariant N of a space cubic K and a linear complex L. The author establishes a connection with the well-known comitants of a space cubic. He shows that when $C=0$, i.e., when the complex L is singular, the equation $N=0$ represents the cubic K in line coordinates. Again, in general the tangents at 4 points on K belong to the complex. The parameters of these points are the roots of a quartic with invariants i, j and discriminant $R=j^2-\frac{1}{6}i^3$. The author expresses these in terms of the invariants given by Primrose [loc. cit.]. Following Lehnen [Dissertation, Bonn, 1921] he remarks on a more convenient method of obtaining the comitants of K by considering the symbolic ground form

$$F=(Au)\alpha_t^3=(A_1u_1+A_2u_2+A_3u_3+A_4u_4)(\alpha_1t_1+\alpha_2t_2)^3,$$

where u_i are plane coordinates and t_1, t_2 binary parameters,

together with a conjugate form giving the same cubic as a plane locus. *D. E. Littlewood* (Bangor).

Convex Domains, Extremal Problems

Kieffer, Lucien. Sur les solides convexes limités par les cônes de révolution circonscrits aux angles solides d'un polyèdre régulier. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. **19**, 221–234 (1950).

The author solves the problem of determining the surface and volume of the convex solid bounded by cones of revolution circumscribing the corners of a Platonic solid. He finds, for instance, that in the case of the unit cube the volume is $3-2\log 2$. *H. S. M. Coxeter* (Toronto, Ont.).

Abe, Yoshibumi, Kubota, Tomio, and Yoneguchi, Hajimu. Some properties of a set of points in Euclidean space. Kōdai Math. Sem. Rep. **1950**, 117–119 (1950).

How few successive applications of the operation of joining points by chords are required to produce from a given set in affine n-space its convex hull? This is answered for arbitrary sets and also for arbitrary sets with at most n components. The first result and also essentially the second, along with further references, may be found in Bonnesen and Fenchel, Theorie der Konvexen Körper [Springer, Berlin, 1934]. *W. Gustin* (Princeton, N. J.).

Pogorelov, A. V. On convex surfaces with regular metric. Amer. Math. Soc. Translation no. **43**, 7 pp. (1951).

Translated from Doklady Akad. Nauk SSSR (N.S.) **67**, 791–794 (1949); these Rev. **11**, 201.

Stoker, J. J. On the uniqueness theorems for the embedding of convex surfaces in three-dimensional space. Comm. Pure Appl. Math. **3**, 231–257 (1950).

Suppose the regular closed convex surface X is the image of the sphere under a three times continuously differentiable topological mapping. The Gauss curvature K of X is assumed to be positive everywhere. The author shows that X is uniquely determined within an orthogonal transformation by each of the following requirements: (1) The first fundamental tensor of X is given as a function on the sphere. (2 & 3) K, respectively the sum of the principal radii of curvature, is prescribed as a function on the spherical image of X. (4) In addition, a proof of Liebmann's theorem on the rigidity of the surfaces X is given. The uniqueness proof in the case of (3) is particularly simple. It uses only the fact that a harmonic function has no extremum in the interior of its domain of regularity. The other proofs are based on a theorem of Hilbert concerning the uniqueness of the solution of a certain linear elliptic differential equation on the sphere. The uniqueness of X in the cases (2) and (4) is proved by straightforward applications of this theorem. In the case of (1), an additional assumption on the regularity of a certain auxiliary surface has to be made, and some properties of the latter have to be derived before Hilbert's theorem can be applied. The four uniqueness proofs are extended to surfaces with a finite number of holes. These holes are assumed to be such that they can be filled in with plane convex domains. After this filling in, X would have to satisfy the original assumptions except that K can now vanish. The inclusion of the relevant facts on support functions and of a proof of Hilbert's theorem makes this paper self-contained. A detailed introduction contributes to its readability. *P. Scherk* (Saskatoon, Sask.).

Hadwiger, Hugo. Zur Minkowskischen Dimensions- und Massbestimmung beschränkter Punktmengen des euklidischen Raumes. Math. Nachr. **4**, 202–212 (1951).

For a bounded set A in E^k denote by A_ρ the union of all open spheres with radius ρ and center in A, and denote by $V(A_\rho)$ the measure of A_ρ. With $q_\tau(A,\rho)=\rho^{\tau-\kappa}V(A_\rho)$, the greatest lower bound μ of the numbers τ for which $\lim_{\rho\to 0+}q_\tau(A,\rho)=0$ is called the Minkowski dimension of A. If $\omega_\kappa=\pi^{\frac{1}{2}\kappa}\Gamma^{-1}(1+\frac{1}{2}\kappa)$ (the volume of the unit sphere for integral κ), put $M_\mu(A)=\limsup_{\rho\to 0+}\omega_{\kappa-\mu}^{-1}q_\mu(A,\rho)$ and $m_\mu(A)=\liminf\omega_{\kappa-\mu}^{-1}q_\mu(A,\rho)$. The following theorem is proved: Given a real μ with $0<\mu<\kappa$ and a pair of numbers α, β with $0<\alpha\leqq\beta<\infty$, there exists a bounded set A in E^k with Minkowski dimension μ and $m_\mu(A)=\alpha$, $M_\mu(A)=\beta$.
H. Busemann (Los Angeles, Calif.).

Hadwiger, H. Minkowskische Addition und Subtraktion beliebiger Punktmengen und die Theoreme von Erhard Schmidt. Math. Z. **53**, 210–218 (1950).

Let Z be an arbitrarily chosen point of Euclidean k-space R_k. Every point P of R_k is described by the vector from Z to P. Given two points sets $A=\{\mathfrak{a}\}$ and $B=\{\mathfrak{b}\}$ in R_k, denote the complement of A by A^* and let $\bar{A}=\{-\mathfrak{a}\}$ be the set symmetric to A with respect to Z. "Minkowski addition" and "subtraction" of A and B yield the sets $A+B$ of all the $\mathfrak{a}+\mathfrak{b}$'s and $A-B$ of all those $\mathfrak{b}$'s for which $\mathfrak{b}+\mathfrak{b}\subset A$ for every $\mathfrak{b}$. The following rules may be of interest:

$$A-B=(A^*+\bar{B})^*;\quad A+B=(A^*-\bar{B})^*,$$
$$(A+B)+C=A+(B+C);\quad (A-B)-C=A-(B+C);$$
$$(A-B)+B\subseteq A\subseteq(A+B)-B.$$

From now on, let A and B be nonempty, closed, and bounded. Whenever $A-B$ occurs, it is also assumed to be nonempty. Let Ω be a $(k-1)$-plane through Z. The symmetrization $S(A)$ of A with respect to Ω is defined as follows: For each straight line $a\perp\Omega$ that intersects A, let $S(Aa)$ be the closed segment of a symmetric with respect to Ω whose length is equal to the one-dimensional Lebesgue measure of Aa. Then $S(A)$ is the union of all these segments $S(Aa)$. It is easy to prove that (1) $S(A+B)\supseteq S(A)+S(B)$, $S(A-B)\subseteq S(A)-S(B)$. Let $r(A)$ denote the radius of a sphere equal in volume to the k-dimensional Lebesgue measure of A. The main result of this paper reads (2) $r(A+B)\geqq r(A)+r(B)$, $r(A-B)\leqq r(A)-r(B)$. An immediate corollary of (2) is the isoperimetric inequality for the outer and inner relative surfaces of A

$$(3)\quad F_{\pm B}(A)=\liminf_{\rho\to\pm 0}\frac{V(A+\rho B)-V(A)}{\rho}\geqq k\cdot V(A)^{1-1/k}V(B)^{1/k}$$

$[\rho B=\{\rho\mathfrak{b}\}]$. The proof of (2) is based on the study of the closure of the family of all the pairs of sets that can be constructed out of A, B through a finite number of symmetrizations.

Some special cases of the above concepts and results were studied by E. Schmidt [Math. Nachr. **1**, 81–157 (1948); **2**, 171–244 (1949); these Rev. **10**, 471; **11**, 534] and others. Let K_ρ denote the sphere of radius $\rho>0$ about Z. Then $A_\rho=A+K_\rho=$outer parallel set of A; $A^{(\rho)}=K_\rho-A$. The relation (1) implies $S(A_\rho)\supseteq(S(A))_\rho$, $S(A^{(\rho)}\subseteq(S(A))^{(\rho)}$. From (2) we obtain $r(A_\rho)\geqq r(A)+\rho$ ["Brunn-Minkowski theorem"] and $r(A^{(\rho)})\leqq\rho-r(A)$ ["Spiegel theorem"]. The

case $+B=+K_1$ of (3) yields the ordinary isoperimetric inequality. Similar formulas follow from (1)–(3) for the inner parallel set $A_{-\rho}=A-K_\rho$ and the inner Minkowski surface $F_{-K_1}(A)$ of A. [On p. 217, formula (10a) is applied in the wrong direction. The author has indicated in a letter to the reviewer how this slip can be corrected.]

P. Scherk (Saskatoon, Sask.).

Hadwiger, H. Verschärfte isoperimetrische Ungleichung für konvexe Rotationskörper mit Spitzen. Math.-Phys. Semesterber. **2**, 98–103 (1951).

Let K be a convex body of revolution in Euclidean 3-space with equatorial radius r, surface area s^2, and volume v^3, whose surface conically meets its axis in the angles ϕ and ψ. Define $A=s^3-6\tau v^3$, $B=(s-2\tau r)^2(s+\tau r)$, $C=\tau(3rs^2-6v^3-4\tau^2r^3)$, $D=\tau^3r^3\{(1-\sin\phi)^2/\sin\phi+(1-\sin\psi)^2/\sin\psi\}$, where $\tau^2=\pi$. Clearly $B\geqq 0$, $D\geqq 0$, and $A=B+C$. The isoperimetric inequality states that $A\geqq 0$ with equality holding if and only if K is a sphere. Bonnesen [Acta Math. **48**, 123–178 (1926)] has improved this by showing that $C\geqq 0$ with equality holding if and only if K is a capsule. The author extends Bonnesen's result by showing that $C\geqq D$ with equality holding if and only if K is a conically capped capsule.

W. Gustin (Princeton, N. J.).

Obrechkoff, Nicolas. Géométrie intégrale hyperbolique. Sbornik Bulgar. Akad. Nauk. **40**, no. 1, 1–46 (1949). (Bulgarian. French summary)

This paper was received by the editors early in 1945. In the meantime equivalent results have been obtained by others, in particular, Santaló [Duke Math. J. **16**, 361–375 (1949); these Rev. **10**, 732] and Vidal Abascal [Integral Geometry on Curved Surfaces, Santiago de Compostela, 1950; these Rev. **11**, 681]. In the Poincaré model of the hyperbolic plane in which the straight lines appear as semicircles in the upper half plane with center $(\alpha, 0)$ and radius r, the density for lines is defined by $g=k\dot r\dot\alpha r^{-2}$, where k is the constant of the hyperbolic geometry considered. Then the length L of a curve C is given by $2L=\int n|g|$, where n is the number of intersections of the line with C. Analogs to Crofton's formulae are derived. For any line g through (x, y), $y>0$, denote by φ the angle which the Euclidean tangent of g at (x, y) forms with the x-axis. Then $A=k^2\dot x\dot y\dot\varphi y^{-2}$ corresponds to Poincare's cinematic density. If C_0 is a fixed curve of length L_0 and C a mobile curve of length L and n the number of their intersections, then $\int nA=4L_0L$. The following hyperbolic analog to Bonnesen's strengthening of the isoperimetric inequality is proved: Let R and r be the radii of the circumscribed and inscribed circles of a convex curve with length L and area F. Then

$$k^2L^2-F^2-4\pi k^2F\geqq 4^{-1}(F^2+4\pi k^2)^2(\tan LR/2k-\tan Lr/2k)^2.$$

Some of these results are extended to 3 dimensions. In the corresponding Poincaré model a hyperbolic plane is a Euclidean hemisphere over the (x, y)-plane with center (α, β) and radius r. The density for planes is then $g=k\dot\alpha\dot\beta\dot\gamma r^{-3}$. The density for a hyperbolic straight line with center α, β, radius R, and angle θ of the Euclidean plane through the line with the x-axis is $G=k^2\dot\alpha\dot\beta\dot R\dot\theta R^{-3}$. If n is the number of intersections of a plane with a curve of length L, then $\int ng=\pi L$, and if n is the number of intersections of a line with a surface of area F, then $\int nG=F$. In the last part some results of integral Möbius geometry are derived by the method used for the hyperbolic plane. *H. Busemann.*

Algebraic Geometry

Chatelet, François. Sur la réalité des courbes de genre un. Ann. Fac. Sci. Univ. Toulouse (4) **11** (1947), 75–92 (1949).

A plane algebraic curve (C) of genus one is a birational transform of a plane cubic (w), $y^2=x^3+px+q$, with $4p^3+27q^2\neq 0$. One asks: (1) Can a given (C) with real coefficients be obtained from a (w) with real coefficients; and (2) can this always be done by a transformation with real coefficients? By manipulating transformations the author answers the first question in the affirmative and the second in the negative. It is shown further that any (C) with real coefficients is obtainable by a transformation with real coefficients from one and only one of the following:

$$\left.\begin{array}{r} y^2=x^3+px+2 \\ y^2=x^3+px-2 \end{array}\right\} \quad \text{for} \quad p\neq -3;$$

$$\left.\begin{array}{r} y^2+x^4+6x^2-4p-3=0 \\ y^2+x^4-6x^2-4p+3=0 \end{array}\right\} \quad \text{for} \quad p<-3;$$

$$y^2=x^3+x; \quad y^2=x^3-x; \quad y^2+x^4+4=0.$$

R. J. Walker (Ithaca, N. Y.).

✶Godeaux, Lucien. Correspondances entre deux courbes algébriques. Mémor. Sci. Math., no. 111. Gauthier-Villars, Paris, 1949. 64 pp. 400 francs.

The author, with admirable clarity, gives in about 60 pages an account of the results and methods of the theory of correspondences between algebraic curves in the complex domain. There is no room for proofs, but adequate references are given and the bibliography is complete up to 1940 when this tract was written. The work is in five chapters. The first introduces the Jacobi variety of a curve C of genus p, following a brief summary of the notions of linear system, the canonical system, and the two types of correspondence between the sets of p points of a curve. The next section deals with an involution on C whose sets are represented by the points of a curve Γ, with the relation between linear series on C and on Γ, and with the correspondence between the Jacobi varieties of these curves. The section ends with a discussion of the problem of enumerating the possible birationally distinct curves C containing an involution of order n represented by a given curve Γ with assigned branch points. Chapter III discusses, by algebro-geometric means, correspondences between two algebraic curves, and the effect of such correspondences on linear systems of the two curves, and deals also with the corresponding results for their Jacobi varieties. Correspondences of valency on a single curve are then discussed, followed by the notion of dependence of correspondences and Severi's discovery of the base for correspondences between two (possibly distinct) curves. A number of applications of the general theory are mentioned. The fourth chapter discusses the transcendental theory of correspondences, beginning with the theory of correspondences on a single curve and their relation to the periods of the simple integrals of the first kind. An account is given of Rosati's geometrical interpretation of Hurwitz's formulae, and of his generalisations of the notions of valency. Hurwitz's application of the theory of theta-functions is mentioned but theta-functions are not employed at any place in the text and the author confines himself to what can be done by more elementary methods, and there is no space for an account of the theory of Riemann matrices. The final chapter of the book discusses the contributions of Chisini and Lefschetz to the topological theory of correspondences. The order of presentation followed by the author is based

on the historical one, and the necessity of compressing the work into such small compass doubtless made it inevitable that it be followed. Nevertheless it is a pity that it was not possible to invert the order of the last two chapters so as to use the topological treatment to illuminate the transcendental theory. The subject matter of this tract already forms a very complete and well rounded theory, and nobody who reads it can fail to be impressed by our enormous debt to the masters of the Italian school with first their German and later their American collaborators. The difference in precision and finality between the theory expounded here and what is known even for correspondences between algebraic surfaces, must ensure that this tract will serve both as a guide and a challenge to further efforts.

D. B. Scott (London).

Godeaux, Lucien. Sur le plan double dont la courbe de diramation se compose de trois coniques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 669–671 (1950).

En rapportant les quartiques du plan passant par les points communs aux trois coniques aux hyperplans de l'espace S^5, l'auteur montre que le plan double représente une surface du 8° ordre à sections de genre 5, dotée de trois points doubles coniques et de douze coniques passant 4 par 4 par deux des points doubles. Cette surface se transforme en elle-même par une homographie biaxiale dont l'un des axes est le plan des points doubles. *B. d'Orgeval.*

Salini, Ugo. Trasformazioni puntuali fra due piani π, π_1 in una coppia di punti corrispondenti (O, O') ad jacobiano nullo di caratteristica uno, nel caso che la curva jacobiana di π abbia un punto doppio in O. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 136–176 (1950).

The author studies the properties of one-to-one transformations T^* between two projective planes π, π_1, in the neighborhood of a pair of corresponding points (O, O'), under the assumption that the Jacobian curve in π has an ordinary double point. The paper is divided into two parts; in the first part the author introduces many geometrical configurations by means of which it is possible to find an intrinsic frame of reference in both planes as well as some projective invariants of the transformation. Moreover, the existence of ∞^2 rational transformations, approximating a given T^* up to the neighborhood of the 2d order of the pair (O, O') is proved. In the second part, a subclass of the T^*'s is studied; the main properties of the $\bar{T}^*$'s, for which a geometrical characterization is also given, is that a given $\bar{T}^*$ may be approximated up to the neighborhood of the 2d order of the pair by a birational transformation (of the 3rd order), which is uniquely determined. *V. Dalla Volta.*

Gröbner, Wolfgang. Über den Multiplizitätsbegriff in der algebraischen Geometrie. Math. Nachr. **4**, 193–201 (1951).

This paper is a plea for adoption of the notion of intersection multiplicity which is based on the length of primary ideals: If two varieties are represented by ideals a and b, then the lengths of the primary ideals of the canonical decomposition of $a+b$ would give the multiplicities of the various components of the intersection of the two varieties. The argument in favor of this definition is that it is simple and that it applies to all cases (even when the intersection has components of "too high" a dimension). The author contends that the other definitions, are complicated, apply only to certain cases, and suffer from the uncertainty of the "principle of continuity" on which they are based. This last objection should certainly not be raised against the perfectly rigorous definition proposed by A. Weil [Foundations of Algebraic Geometry, Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; these Rev. **9**, 303; this book is quoted in the bibliography of the paper now being reviewed, but no reference to it is made in the text; as for the reviewer's paper on the subject [Trans. Amer. Math. Soc. **57**, 1–85 (1945); these Rev. **7**, 26], it does not appear even in the bibliography]. The author seems to consider that the only possible objection against the definition he proposes is that it does not lead to Bézout's theorem in the general case. Much more serious, it seems to the reviewer, is the fact that the author's definition cannot have a certain number of simple properties, such as associativity of the intersections, or the formulas for projections of intersections on a product variety, properties which can be proved to determine entirely the intersection multiplicities as defined by A. Weil or the reviewer. *C. Chevalley* (New York, N. Y.).

Differential Geometry

van der Waag, Eduard Johannes. Sur quelques notions fondamentales de la géométrie différentielle. C. R. Acad. Sci. Paris **231**, 1026–1027 (1950).

(I) C désigne une courbe de l'espace euclidien E_3, O un point fixe sur C distinct des extrémités, A et B deux points sur C distincts. L'arc de C est dit équivalent à la corde en O au sens ordinaire si le rapport arc AB/corde AB tend vers 1 pour $A=O$ et $B\to O$. L'équivalence est appelée uniforme si ce rapport$\to 1$ pour $A\to O$ et $B\to O$ de manière quelconque. Théorèmes: Une condition nécessaire et suffisante pour que la courbe C soit rectifiable au voisinage de O et donne lieu en O à l'équivalence uniforme, est l'existence d'une représentation paramétrique $\mathbf{p}(t)$ pour un voisinage de O, telle que $\lim\,(|\mathbf{p}(t'')-\mathbf{p}(t')|/|t''-t'|)=1\neq 0$. Pour que la courbe plane rectifiable C admette en O l'équivalence ordinaire, il faut et il suffit qu'il existe une représentation paramétrique $\mathbf{p}[\alpha, \beta]$ d'un segment I de C contenant O, le paramètre de O étant$=0$, telle que (i) $\lim_{t\to 0}|\mathbf{p}(t)|/|t|=\lambda>0$; (ii) les dérivés de $\mathbf{p}$ soient bornés sur $[\alpha, \beta]$; (iii) $\mathbf{p}'$ existe presque partout sur $[\alpha, \beta]$ et $\lim|\mathbf{p}'(t)|=\lambda$. (II) L'auteur donne huit définitions de plan osculateur pour C en O [cf. Pauc, Les méthodes directes en géométrie différentielle, Actualités Sci. Ind., no. 886, Hermann, Paris, 1941 ; ces Rev. **7**, 67] faisant intervenir des points voisins de O ou des tangentes en des points voisins de O. Voici un exemple de théorème obtenu: Si trois points quelconques A, B, C dans un voisinage droit de O définissent un plan ayant une limite quand A, B, C tendent vers O, C admet en O une demi-tangente à droite; si celle-ci est prise comme axe des x, la courbe à droite de O peut être représentée par $y=f(x)$ et $z=g(x)$. *C. Pauc.*

van der Waag, Eduard Johannes. Sur quelques notions fondamentales de courbure. C. R. Acad. Sci. Paris **231**, 1120–1122 (1950).

[Continuation de la note précédente.] Les définitions sont empruntés au référent [loc. cit]. Théorèmes: Si C admet en O une courbure et un plan osculateur de Alt et si la tangente ordinaire existe dans un voisinage de O, la courbure classique existe. Si C est plane et admet en O une courbure de Alt infinie, un voisinage de O sur C est rectifiable. Si un continu C admet en O une courbure de Gödel finie$=\kappa_0$, pour chaque suite régulière (P_n', O, P_n''), $\lim \kappa(P_n', O, P_n'')=\kappa_0$. Ce

théorème n'est pas exact pour un ensemble quelconque comme l'a affirmé le référent ; il le devient si l'on postule l'existence du plan osculateur de Gödel. Les résultats des deux notes ne sont que partiellement reproduits dans ces analyses. Ils sont formulés avec concision. Aucune démonstration n'est donnée. *C. Pauc* (le Cap).

Haupt, Otto. Über die ebenen Bogen der linearen Ordnung Drei. Acta Univ. Szeged. Sect. Sci. Math. **13**, 153–162 (1950).

An arc $\mathfrak{A}$ is the single-valued continuous image of a segment in the real projective plane. The order (index) of $\mathfrak{A}$ is the maximum (minimum) number of points that $\mathfrak{A}$ has in common with any straight line. The order of a point of $\mathfrak{A}$ is the order of a sufficiently small neighborhood on $\mathfrak{A}$ of that point. The author classifies the arcs $\mathfrak{B}$ of order three. The following features are considered: The index of $\mathfrak{B}$; the number of double points; the number of points in which $\mathfrak{B}$ is met by the straight line through the end-points A, C of $\mathfrak{B}$ and, if $\mathfrak{B}$ has the index zero, their order on AC; the position of small neighborhoods (on $\mathfrak{B}$) of A and C in relation to AC and to the one-sided tangents at A and C; the number and types of the points of order three on $\mathfrak{B}$; the minimum number of convex arcs into which $\mathfrak{B}$ can be decomposed. It follows from this classification that this minimum number is not greater than four, respectively three, if $\mathfrak{B}$ has the index zero, respectively one. The first part of this corollary was proved by Marchaud [Acta Math. **55**, 67–115 (1930)]. Besides the above classification which is stated without proofs, the present paper contains a simple direct proof of Marchaud's result. *P. Scherk* (Saskatoon, Sask.).

Haupt, Otto. Über eine Beziehung zwischen Ordnung und Singularitäten. Math. Nachr. **4**, 81–96 (1951).

The object of this paper is to investigate the underlying topological content of the following result from differential geometry which connects the circular order of a curve with the existence of a special type of singularity. An oval which possesses an osculating circle at each of its points and which has only a finite number of vertices has at most four points in common with any circle if and only if it has at most two common tangents with any circle. The investigation concerns a related but more general problem in which the oval is replaced by a circle $\mathfrak{K}$ of circumference 1 and the system of intersecting circles by a system $\mathfrak{m}$ of finite point sets of $\mathfrak{K}$ called complexes. Let h, k be integers with $3 \leqq k < h$. The complexes of $\mathfrak{m}$ satisfy the following four conditions. (1) If r denotes the number of different points of a complex then $k < r \leqq h$. (2) Exactly one complex of $\mathfrak{m}$ contains k arbitrary points of $\mathfrak{K}$. (3) Each complex depends continuously on any k points which it contains. Provision is made for the case in which the number of elements of a complex varies under continuous deformation of k elements which define it. (4) If $k-1$ points of a complex are fixed and one other point moves continuously in a fixed direction on $\mathfrak{K}$ then all the moving points move in fixed directions so that any two consecutive moving points move in opposite directions. Corresponding to the property that no circle has more than two common tangents with the oval in the original problem is the following property of the complexes called boundedness of concentration. The system $\mathfrak{m}$ is defined to be of bounded concentration if a positive constant b exists for which the following is true. For any complex α in $\mathfrak{m}$, nonoverlapping open arcs $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_t$ of $\mathfrak{K}$ exist, each of length less than b, so that $\mathfrak{K}_\tau$ contains q_τ points of α, $q_\tau \geqq 2$, $1 \leqq \tau \leqq t$. If $q_\tau \leqq k$, $1 \leqq \tau \leqq t$, then $\sum_{1 \leqq \tau \leqq t}(q_\tau - 1) \leqq k-1$, but if at least one $q_\tau > k$ then $t=1$ and $q_1 = k+1$. By means of continuous deformations of the complexes, the author proves that a system $\mathfrak{m}$ of complexes which satisfy the above four conditions is of bounded concentration if and only if the number of points in each complex is $k+1$. As an application of this result a proof of the initial theorem is given. A number of known results of the same character as this theorem are listed. The author indicates a proof of the following with the use of his principal result. If $\mathfrak{C}$ is a closed simple curve in the projective plane whose points have at most local order 3 and for which a tangent is defined then $\mathfrak{C}$ is of order 3 if and only if it has no double tangents. *D. Derry* (Vancouver, B. C.).

Nanjundiah, T. S. Contributions to affine differential geometry. I. J. Roy. Asiatic Soc. Bengal. Sci. **15**, 92–104 (1949).

Servendosi dei metodi della geometria differenziale affine, l'autore studia alcune proprietà delle coniche osculatrici e della normale affine ad una curva piana in un suo punto (ordinario, non di flesso), ritrovando molti dei risultati ottenuti da S. Mukhopadaya [J. Proc. Asiatic Soc. Bengal (N.S.) **4**, 167–178, 391–402, 497–509 (1908)]. La maggior parte dei risultati sono di natura metrica piuttosto che affine (p.es. la considerazione dell'iperbole equilatera osculatrice ad una data curva in un suo punto), e solo scopo dell'autore è di provare che si possono ottenere più direttamente con i suoi metodi. [Osservazione del recensore: Non appare ben chiara l'invarianza per affinità delle quantità che appaiono nelle formule (3.8) e (3.10) dell'autore, giacchè nelle loro espressioni compariscono lunghezze di vettori e prodotti scalari, che non sono invarianti affini.] *V. Dalla Volta.*

Fenchel, Werner. On the differential geometry of closed space curves. Bull. Amer. Math. Soc. **57**, 44–54 (1951).

The present paper, while rather brief and expository in character, is written in such a way as to give a deep insight into the subject. Also it points out a number of unsolved questions, and contains a fairly extensive bibliography. The paper refers to a space curve K, represented by functions of class 4, for which Frenet's formulae $\mathbf{t}' = \kappa\mathbf{n}$, $\mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}$, $\mathbf{b}' = -\tau\mathbf{n}$ are valid in the usual form, but that κ is allowed to vanish and even be negative, the only assumption being that κ and τ may only vanish at a finite number of points and never simultaneously. Then the spherical indicatrices T, B, N, C, i.e. the loci of the points $O+\mathbf{t}$, $O+\mathbf{b}$, $O+\mathbf{n}$, $O+(\kappa\mathbf{b}+\tau\mathbf{t})(\kappa^2+\tau^2)^{-\frac{1}{2}}$ (where O denotes a fixed point), are considered, and their mutual relations are investigated. Next, necessary and sufficient conditions are given for a closed curve T, lying on the unit sphere of centre O, to be the tangent indicatrix of a closed curve K; a more stringent necessary condition for T is obtained when K is knotted. Similar questions are also considered for the spherical indicatrices B and N. *B. Segre* (Rome).

Urban, A. L'équation différentielle des courbes sur V_{n-1} spéciale dans V_n. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 48 (1947), 57–61 (1950).

Résumé of a paper which appeared earlier in Czech [Rozpravy II. Třídy České Akad. **57**, no. 9 (1948); these Rev. **9**, 618].

Hopf, Heinz. Über Flächen mit einer Relation zwischen den Hauptkrümmungen. Math. Nachr. **4**, 232–249 (1951).

This paper considers several questions in the large relating to a surface in Euclidean three-space. Generally speaking,

the surface is required to be analytic, but some results are true under weaker assumptions. First, there is a proof of an extension of a classical theorem of Liebmann: "Among all closed surfaces which are topological spheres, the (metric) spheres are the only ones with constant mean curvature." The proof depends upon a classical theorem of Poincaré and the theorem: "Let F be a surface element with constant mean curvature which is not part of a sphere or a plane, and let P be an umbilic on F. Then P is isolated and has a negative index with respect to the family of lines of curvature corresponding to the greater of the two principal curvatures." Second, the author turns to the so-called "H-Satz, K-Satz, and HK-Satz" of Liebmann. These all consider surfaces whose principal curvatures satisfy a relation $W(k_1, k_2)=0$; i.e. the surfaces are Weingarten surfaces. The author raises the question: "What are the closed Weingarten surfaces?" The only ones now known have genus 0 or 1, but there is no reason why W-surfaces of higher genus may not exist. By considering the W-diagram of a surface in the (k_1, k_2)-plane, he proves: "Let a curve C be given in the (k_1, k_2)-plane which is differentiable at its points of intersection with the line $k_1=k_2$; and let $dk_1/dk_2=\kappa$ at these points. If all values of κ are negative, C is not the W-diagram of a closed analytic W-surface of genus zero; if all values of κ are positive, C is not the W-diagram of a closed analytic W-surface of genus at least 2. If κ does not have one of the values: $0, \infty, -1, 3^{\pm 1}, 5^{\pm 1}, \cdots, (2m+1)^{\pm 1}, \cdots$, then C is not the W-diagram of a closed analytic W-surface of genus $\neq 1$." This implies the following generalization of the H-Satz and the K-Satz: "Let $W(k_1, k_2)$ be differentiable and symmetric in k_1 and k_2; and let the function $f(k)=W(k, k)$ have no multiple zeros. Then every analytic surface of genus zero on which $W(k_1, k_2)=0$ holds is a metric sphere." Finally, consider a linear relation L $(k_2=ck_1+d)$ between k_1 and k_2. It is proved that a closed analytic surface of genus zero on which L holds is either a metric sphere $(c=1, d=0)$; or has the property: it has exactly two points at which $k_1=k_2=0$ and no other umbilics (hence $d=0$); at other points $K>0$; $c=(2m+1)^{\pm 1}$ with $m>0$. All these surfaces actually exist as is shown by examples. *C. B. Allendoerfer.*

Wu, George. A note on the asymptotic chord quadrics of a surface. Acad. Sinica Science Record **2**, 345–350 (1949).

Let $Q^{(u)}$ and $Q^{(v)}$ be the Bompiani asymptotic chord quadrics [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 9, 288–294 (1929)] associated with a curve C on a surface S through its point P. Let ω be the osculating plane of C at P. The locus of intersections $\omega Q^{(u)}$ $(\omega Q^{(v)})$ for all C's with the same tangential line at P is a quadric $\bar{Q}^{(u)}$ $(\bar{Q}^{(v)})$. If C approaches either one of the asymptotic lines $u=c$ or $v=c$ both quadrics approach the same quadric in the Darboux pencil. Let π be the plane of the residual conic intersection of $\bar{Q}^{(u)}$ and $\bar{Q}^{(v)}$. All three planes π corresponding to the directions of Segre or Darboux are concurrent in the same canonical line $c=(-\frac{2}{3})$ or $c=(-1)$. Some other theorems of this kind are proved. The proofs are based on a straightforward computation. *V. Hlavatý.*

Löbell, Frank. Zur Frage der Vertauschbarkeit geodätischer Richtungsableitungen. Math. Ann. **122**, 152–156 (1950).

In a previous paper [same Ann. 121, 427–445 (1950); these Rev. 11, 686], the author introduced differentiation of line element functions, i.e. functions on a surface which depend on a point and a tangent direction. The present paper discusses the iteration of this directional differentiation, and the result of interchanging the order of differentiation. The resulting differential operators are written in an invariant form involving the Gaussian curvature of the surface. *S. B. Jackson* (College Park, Md.).

Charrueau, André. Sur les systèmes linéaires de complexes linéaires. C. R. Acad. Sci. Paris **232**, 144–145 (1951).

Charrueau, André. Sur les systèmes linéaires de complexes linéaires. C. R. Acad. Sci. Paris **232**, 202–204 (1951).

L'auteur démontre les propriétées bien connues des systèmes linéaires de complexes linéaires dans l'espace à 3 dimensions à l'aide de correspondance suivante: À chaque complexe du système $\sum_{i=0}^{n}\lambda_i\varphi_i=0$, $n\leqq 4$, où $\varphi_i(p_1, \cdots, p_6)=0$ sont les équations en coordonées plückeriennes des $n+1$ complexes de base du système, on fait correspondre un point de coordonnées $\lambda_0, \lambda_1, \cdots, \lambda_n$ de l'espace projectif à n-dimensions. Les cas à considérer sont ceux deux $n=1, 2, 3$, ou 4. Dans le cas $n=4$ on a spécialement les théorèmes pour les points figuratifs des complexes conjugués (polaires) du complexe donné par rapport aux complexes non spéciaux du système à 4 paramètres. *F. Vyčichlo* (Prague).

Vančura, Zdeněk. Les congruences de Lie-sphères (L-sphères). Acta Fac. Nat. Univ. Carol., Prague no. **194**, 20–28 (1948). (French. Czech summary)

The set of all Lie spheres in a linear three-space E is mapped on a four-dimensional quadric (Q) in a five-dimensional projective linear space (P). A congruence S of Lie-spheres is mapped on a surface (S) on (Q). The surface (S) possesses invariants on (Q) in (P) and these invariants have their interpretation for the congruence S in E [cf. Hlavatý, Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd **1941**; Rozpravy II. Třídy České Akad. **51** (1941); **52** (1942); these Rev. **7**, 483; **9**, 64]. The author continues the reviewer's investigations of S. First he finds necessary and sufficient conditions (expressed in terms of the mean and Gaussian curvatures of (S) or in terms of mean surfaces) that the second osculating space of (S) be m-dimensional ($m=3, 4, 5$, the case $m=4$ being an analogy to Weingarten line congruence). Then he investigates both couples of principal surfaces of S and this investigation leads to the relationship between the mean curvatures and the "parameters of distribution" $\overset{x}{B}_{cd}v^cv^d/a_{cd}v^cv^d$ (where a_{cd}, $\overset{x}{B}_{cd}$ stand for the metric tensor and the xth tensor of S $(x=2, 3)$ and v^a denotes an arbitrary surface of S). It turns out that this relationship is a generalization of the Enneper relationship in the theory of surfaces. The last two chapters are devoted to the investigation of the base of a Lie sphere in S, of its "focal points" and "focal planes" and to applications. *V. Hlavatý* (Bloomington, Ind.).

Tomonaga, Yasuro. On Betti numbers of Riemannian spaces. J. Math. Soc. Japan **2**, 93–104 (1950).

The principal result of this paper is that if M is a compact orientable Riemannian manifold (a) which is symmetric $(R^i_{jkl,m}=0)$; (b) for which

$$\{p(p-1)R_{ijkl}g_{st}+g_{kl}(2pR_{ijts}-pR_{ij}g_{st}-R_{st}g_{ij})\}\xi^{lis}\xi^{kjt}$$

is negative for all nonzero tensor fields ξ^{ijk} skew-symmetric in the first two indices, then every harmonic tensor $\xi_{i_1\cdots i_p}$ on M has covariant derivative zero. The theory of harmonic tensors is used to establish certain topological properties of

compact orientable Riemannian manifolds; those which follow from the existence of vector fields on M with zero covariant derivatives are contained in more general theorems announced by Lichnerowicz [C. R. Acad. Sci. Paris **231**, 1280–1282, 1413–1415 (1950); **232**, 146–147 (1951); these Rev. **12**, 535, 536] and H. Guggenheimer [ibid. **232**, 470–472 (1951); these Rev. **12**, 535]. The proof that the mth Betti number of a compact symmetric orientable Riemannian manifold is always positive for $2 \leqq m < \frac{1}{2}n$ appears to depend on the unproved assertion that certain differential forms are not zero. Applications are made to the problem of infinitesimal collineation on a Riemannian manifold and to obtain certain results due to Bochner [Ann. of Math. (2) **49**, 379–390 (1948); these Rev. **9**, 618].

W. V. D. Hodge (Cambridge, England).

Reeb, Georges. Sur une propriété globale des variétés minima d'un espace de Cartan. C. R. Acad. Sci. Paris **232**, 1279–1280 (1951).

By a Cartan space of dimension n is meant one in which there is given a multiple integral of order $n-1$. At each point this gives rise to a norm in the space of covariant $(n-1)$-vectors, of which those of norm 1 form a submanifold W_{2n-1} of dimension $2n-1$. In W_{2n-1} there is naturally defined a differential form Ω of degree n. Let Γ be the set of completely decomposable $(n-1)$-vectors of W_{2n-1} whose interior products with Ω are zero. The author proves the theorem: There is no compact orientable manifold W_n of n dimensions in W_{2n-1}, which is not tangent to any element of Γ.

S. Chern (Chicago, Ill.).

Kuiper, N. H. Einstein spaces and connections. I. Nederl. Akad. Wetensch., Proc. **53**, 1560–1567 = Indagationes Math. **12**, 505–512 (1950).

Kuiper, N. H. Einstein spaces and connections. II. Nederl. Akad. Wetensch., Proc. **53**, 1568–1576 = Indagationes Math. **12**, 513–521 (1950).

The author studies the properties of normal conformal and normal projective connections determined uniquely by an Einstein metric, i.e., by a Riemannian metric such that the Ricci tensor is proportional to the fundamental tensor. In section 1, the author studies the conformal group by a projective treatment (the conformal group being regarded as a subgroup of the general projective group P^{n+1} leaving invariant a quadratic hypersurface S^n in an $(n+1)$-dimensional projective space), and especially the subgroup which leaves invariant a point ψ on, inside, or outside S^n. If ψ is on S^n, then there exists a system of coordinates such that the subgroup G_0, leaving ψ invariant, leaves the expression $ds^2 = \sum_{a=1}^n (y^a)^2$ invariant. If ψ is inside or outside S^n, then there exists a system of coordinates such that the subgroup G_K, leaving ψ invariant leaves $K \cdot \sum_{a=1}^n (y^a)^2 + 4 = 0$ invariant. In section 2, using the terminology of the theory of fibre spaces, the author gives a strict definition of a space with conformal connection in the sense of É. Cartan. The conformal connection is supposed to be defined by the equations of the form $dy^a + \omega_i^e dx^i Y_e y^a = 0$, Y_e being the operators of infinitesimal conformal transformations. The g_{ij} given and ω_i^a appearing here are connected by the relation $g_{ij} = \sum_{a=1}^n \omega_i^a \omega_j^a$.

In section 3, the author states first that the curvatures of the G_K-connection and G_0-connection are related by

$$\Omega^i{}_{jkl}(K) = \Omega^i{}_{jkl}(0) - K(\delta_k{}^i g_{jl} - \delta_l{}^i g_{jk}).$$

He then proves the main theorem: The normal conformal connection of an Einstein space is the conformal abstractum of the G_K-connection of the space. From this main theorem, he deduces theorems such as: A conformally flat Einstein space is a space of constant curvature [Schouten and Struik, Amer. J. Math. **43**, 213–216 (1921)]; the normal conformal P^{n+1}-connection of an Einstein space has a covariant constant point, which lies outside, on, or inside the invariant S^n, according as the scalar curvature K is $<$, $=$, or >0, respectively; if the normal conformal P^{n+1}-connection of a Riemannian space has a covariant constant point interior, on, or exterior to the covariant constant S^n, then the Riemannian space is conformal to an Einstein space with positive, zero, or negative scalar curvature, respectively [Sasaki, Jap. J. Math. **18**, 615–622, 623–633, 791–795 (1943); these Rev. **7**, 330]. In section 4, the author treats the normal projective connection determined uniquely by an Einstein metric. The main theorem is: The normal projective connection of an Einstein space of scalar curvature K is the projective abstractum of the G_K-connection. From this he deduces theorems such as: The normal projective connection of an Einstein space has a covariant constant hypersurface of the second degree [Sasaki and Yano, Tôhoku Math. J. (2) **1**, 31–39 (1949); these Rev. **11**, 398].

K. Yano (Princeton, N. J.).

Egorov, I. P. On groups of motion of spaces with general asymmetrical affine connection. Doklady Akad. Nauk SSSR (N.S.) **73**, 265–267 (1950). (Russian)

Let L_n be an n-dimensional space with an asymmetrical affine connection whose object of translation is

$$\Lambda^\alpha{}_{\beta\gamma}(x^1, \cdots, x^n).$$

The set of infinitesimal transformations of the group of motions relative to v^σ $(u_\tau{}^\sigma = v_\tau{}^\sigma)$ is given by the relations (1) $v_\beta{}^\alpha = u_\beta{}^\alpha$, (2) $u^\alpha{}_{\beta,\gamma} = R^\alpha{}_{\beta\gamma\delta} v^\delta$,

$$(3) \qquad v^\sigma \Omega^\alpha{}_{\beta\gamma,\sigma} + u_\beta{}^\sigma \Omega^\alpha{}_{\sigma\gamma} + u_\gamma{}^\sigma \Omega^\alpha{}_{\beta\sigma} - u_\sigma{}^\alpha \Omega^\sigma{}_{\beta\gamma} = 0.$$

Here the torsion tensor satisfies the following relations:

$$(4) \qquad \begin{aligned} \Omega^\alpha{}_{\beta\gamma} &= \Lambda^\alpha{}_{[\beta\gamma]}, \\ \Omega^\alpha{}_{\beta\gamma} &= \delta_\beta{}^\alpha \Omega_\gamma - \delta_\gamma{}^\alpha \Omega_\beta, \end{aligned}$$

$R^\alpha{}_{\beta\gamma\delta}$ is the curvature tensor of the space whose object $\Lambda^\alpha{}_{(\beta\gamma)}$ determines also a covariant derivative. A necessary and sufficient condition that L_n be a space with an asymmetrical affine connection is: In every coordinate system (5) $\Omega^\alpha{}_{\beta\gamma} = 0$. J. P. Egorov proved [same Doklady (N.S.) **64**, 621–624 (1949); these Rev. **10**, 739] that the maximum order of a complete group of motions in a space L_n with an asymmetrical connection is n^2. A space L_n is called a space with a general asymmetrical connection if the torsion tensor $\Omega^\alpha{}_{\beta\gamma}$ has a general structure. For spaces of this type the author proves by an algebraic analysis of the relations (1), (2), (3) the following theorem: The maximum order of a complete group of motions is $n^2 - 2n + 6$. He adds (without proof) that any symmetrical connection $\Lambda^\alpha{}_{(\beta\gamma)}$ of all spaces with the connection $\Lambda^\alpha{}_{\beta\gamma}$ which have complete groups of motions of order $n^2 - 2n + 6$ is necessarily projectively Euclidean.

F. Vyčichlo (Prague).

Lopšic, A. M. On the theory of a hypersurface in $(n+1)$-dimensional equiaffine space. Trudy Sem. Vektor. Tenzor. Analizu. **8**, 273–285 (1950). (Russian)

A hypersurface is defined by the vector equation $r = r(\eta^1, \cdots, \eta^n)$. Then $r_{ik} = \partial^2 r / \partial\eta^i \partial\eta^k$ can be written in the form $r_{ik} = \Gamma^\alpha{}_{ik} r_\alpha + \alpha_{ik} n$, where n is some vector outside of the tangent hyperplane. Then, if we write λ_{ik} for the alternation

$(r_{ik}r_1\cdots r_n)$, $\lambda=\det(\lambda_{ik})$, then n is normalized by the condition $\alpha_{ik}=\lambda_{ik}|\lambda|^{-1/(n+2)}=\gamma_{ik}$. Then $(nr_1\cdots r_n)=|\gamma|^{\frac{1}{2}}$, $\gamma=\det(\gamma_{ik})$. The γ_{ik} is considered the metrical tensor. We further impose the condition that n be selected such that $n_i=\beta_i^{\ \mu}r_\mu$. In this case n is the affine normal, and coincides with the affine normal of Blaschke [Vorlesungen über Differentialgeometrie . . ., v. 2, Springer, Berlin, 1923]. It determines the $\Gamma_{ik}^{\ \alpha}$ uniquely as the coefficients of a symmetrical connection in the hypersurface. Let $A^i_{\ kl}=\Gamma^i_{\ kl}-\bar{\Gamma}^i_{\ kl}$, where the $\bar{\Gamma}^i_{\ kl}$ belong to the metrical tensor γ_{ik}. The tensor $A_{ikl}=A^\mu_{\ ik}\gamma_{\mu l}$ is symmetrical in ik and in kl. Now stress is laid on two theorems. The first theorem establishes the necessary and sufficient conditions that given $\Gamma^i_{\ kl}$ and γ_{ik} can be taken as the coefficients of connection and the metrical tensor of a hypersurface. The second theorem establishes the restrictions on the tensors γ_{ik} and A_{ikl} such that there exists a hypersurface for which γ and A have the given interpretation. The γ and A can be taken as the first and second fundamental tensor of the hypersurfaces. The author refers to an earlier paper [same Trudy 6, 365–419 (1948)].

D. J. Struik (Cambridge, Mass.).

Lopšic, A. M. On the theory of a surface of n dimensions in an equicentroaffine space of $n+2$ dimensions. Trudy Sem. Vektor. Tenzor. Analizu. 8, 286–295 (1950). (Russian)

A hypersurface S_n is defined by the vector equation $OM\equiv r=r(\eta^1, \eta^2, \cdots, \eta^n)$. The vector r has its beginning at the fixed point O of the space E_{n+2}; we consider a section of S_n where r does not lie in the tangent hyperplane T_n. We take r as first normal at M and another vector n through M, not in T_n and not along r, as second normal vector. Then we can write

$$(1)\qquad r_{ik}=\partial^2 r/\partial\eta^i\partial\eta^k=\Gamma^\alpha_{\ ik}r_\alpha+\sigma_{ik}r+\alpha_{ik}n.$$

The n is normalized by the equations

$$\alpha_{ik}=\lambda_{ik}|\lambda|^{-1/(n+2)}=\gamma_{ik},\quad \lambda=\det|\lambda_{ik}|,\quad \lambda_{ik}=(r_{ik}r_1\cdots r_n\bar{n}),$$
$$(nr_1r_2\cdots r_nr)=|\gamma|^{\frac{1}{2}},\quad \gamma=\det|\gamma_{ik}|.$$

The γ_{ik} is considered as the "first interior tensor." We further require that (2) $n_i=\beta_i^{\ \mu}r_\mu+\beta_i r$. This determines a field of second normal vectors $(n, \bar{n})$, and related by $n=\bar{n}-\xi^\alpha r_\alpha-\xi r$ and for which always $\sigma_{\alpha\beta}\gamma^{\alpha\beta}=0$, where $\gamma^{\alpha\beta}$ is the contravariant tensor belonging to $\gamma_{\alpha\beta}$: $\gamma_{\alpha\beta}\gamma^{\alpha\beta}=n$. The $\Gamma^\alpha_{\ ik}$ and σ_{ik} arising from the choice of n form the "interior connection" and the "second interior tensor" of S_n. Now the necessary and sufficient conditions are established under which a given set $\Gamma^i_{\ kl}$, σ_{ik} and γ_{ik} (all functions of the η, $\det|\gamma|\neq 0$) can be considered as determining the interior connections and the interior tensors of a hypersurface S_n in the given E_{n+2}. For this purpose the following equations are derived by alternating differentiation from (1) and (2):

$$\rho^i_{\ klm}=\beta^i_{\ [l}\gamma_{m]k}+\delta^i_{\ [l}\sigma_{m]k},$$
$$\gamma_{i[k/l]}=0,\quad \sigma_{i[k/l]}+\gamma_{i[k}\beta_{l]}=0,\quad \beta^k_{\ [l/m]}=\delta^k_{\ [l}\beta_{m]},$$
$$\beta^\alpha_{\ [l}\gamma_{m]\alpha}=0,\quad \beta^\alpha_{\ [l}\sigma_{m]\alpha}+\beta_{[l/m]}=0,$$

where $\rho^i_{\ klm}$ is the curvature tensor belonging to the $\Gamma^i_{\ kl}$, and the covariant derivatives are also taken with respect to the $\Gamma^i_{\ kl}$. It is now shown that the fourth, as well as the sixth, of these equations is a consequence of the first three equations, if $n\neq 2$. These results are analogous to those of T. Y. Thomas [Acta Math. 67, 169–211 (1936)].

D. J. Struik (Cambridge, Mass.).

Gurevič, G. B. Canonization of a pair of bivectors. Trudy Sem. Vektor. Tenzor. Analizu. 8, 355–363 (1950). (Russian)

From the two given bivectors $v_{\alpha\gamma}$, $\bar{v}^{\alpha\gamma}$ the following tensors are formed: $L_\alpha^{\ \beta}=v_{\alpha\gamma}\bar{v}^{\beta\gamma}$, $T_\alpha^{\ \beta}=L_\alpha^{\ \beta}-\lambda\delta_\alpha^{\ \beta}$, $(T^2)_\alpha^{\ \beta}=T_\gamma^{\ \beta}T_\alpha^{\ \gamma}$, $\cdots$, $(T^k)_\alpha^{\ \beta}=(T^{k-1})_\gamma^{\ \beta}T_\alpha^{\ \gamma}$. Then, for all k, $(T^k)_\alpha^{\ \gamma}v_{\gamma\beta}=v_{\alpha\gamma}(T^k)_\beta^{\ \gamma}$ and similar formulas hold for $\bar{v}$. Let $\lambda_1, \cdots, \lambda_n$ be different characteristic numbers of L. Then, if $\bar{u}^\alpha$ belongs to λ_1 (multiplicity $\leqq h$), we have: $T^h\bar{u}=0$, and $T^hp=0$, where $p_\alpha=v_{\gamma\alpha}\bar{u}^\gamma$. If q_α similarly belongs to $\lambda_2\neq\lambda_1$ then $(q\bar{u})=q_\alpha\bar{u}^\alpha=0$. Hence, if $\bar{u}_1$ and $\bar{u}_2$ lie in the invariant spaces belonging to different λ, then $(v_{\alpha\beta})(\bar{u}_1)^\alpha(\bar{u}_2)^\beta=0$. We can proceed similarly for $\bar{v}$. Let now $T^{k+1}=0$, $T^k\neq 0$. If we select $\bar{a}_0$ such that $T^k\bar{a}_0\neq 0$ and write $T^i\bar{a}_0=\bar{a}_i$, $(p_i)_\alpha=v_{\gamma\alpha}(\bar{a}_i)^\gamma$, then, since $p_k\neq 0$, there exists a vector $\bar{b}_0$ such that $(p_k\bar{b}_0)=1$. If we now write $T^i\bar{b}_0=\bar{b}_i$, $(q_i)_\alpha=v_{\gamma\alpha}(\bar{b}_i)^\gamma$, then the vectors $\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_k; \bar{b}_0, \bar{b}_1, \cdots, \bar{b}_k$ as well as $-q_k, -q_{k-1}, \cdots, -q_0$; $p_k, p_{k-1}, \cdots, p_0$ form two reciprocal bases for the space S spanned by $\bar{a}_i$, $\bar{b}_i$. Now the following theorem is proved. The space S can be represented as the straight sum of spaces S_i such that v, $\bar{v}$, and L are decomposed into parts of which each lies in one of the S_i, and each of the S_i corresponds to two chains of vectors $(\bar{a}\bar{b})$ and (pq). If $\lambda\neq 0$, then the parts of v and $\bar{v}$ which lie in the corresponding S_i can be given either the form

$$(1)\qquad [p_0q_k]+[p_1q_{k-1}]+\cdots+[p_kq_0],$$
$$(2)\qquad \lambda\{[\bar{a}_0\bar{b}_k]+[\bar{a}_1\bar{b}_{k-1}]+\cdots+[\bar{a}_k\bar{b}_0]\}$$
$$+[\bar{a}_1\bar{b}_k]+[\bar{a}_2\bar{b}_{k-1}]+\cdots+[\bar{a}_k\bar{b}_1].$$

or

$$(3)\qquad \lambda\{[p_0q_k]+[p_1q_{k-1}]+\cdots+[p_kq_0]\}$$
$$+[p_1q_k]+[p_2q_{k-1}]+\cdots+[p_kq_1],$$
$$(4)\qquad [\bar{a}_0\bar{b}_k]+[\bar{a}_1\bar{b}_{k-1}]+\cdots+[\bar{a}_k\bar{b}_0].$$

If $\lambda=0$, then these parts take either the form (1), (2) with $\lambda=0$, or (3), (4) with $\lambda=0$, or they take the form

$$[p_0q_k]+[p_1q_{k-1}]+\cdots+[p_{k-1}q_1],$$
$$[\bar{a}_1\bar{b}_k]+[\bar{a}_2\bar{b}_{k-1}]+\cdots+[\bar{a}_k\bar{b}_1].$$

These results can be used to derive the classification for the case of two covariant bivectors, and we obtain the results of Weierstrass and Kronecker. The number k, belonging to the subspace with $\lambda=0$, is found to be the arithmetical invariant of Kronecker.

D. J. Struik (Cambridge, Mass.).

Gurevič, G. B. On some affinors connected with trivectors of the eighth rank. Trudy Sem. Vektor. Tenzor. Analizu. 8, 296–300 (1950). (Russian)

In a preceding paper [same Trudy 6, 89–104 (1948)] two concomitant tensors $h^k_{\ ij}$, c_{ij}, belonging to the trivector w_{ijk} of rank eight, were introduced, as well as the tensor $H_i^{\ j}=h^j_{\ i\alpha}\bar{l}^\alpha$. This tensor H has eight invariant directions, of which three belong to the characteristic number $\lambda_1=-1$, four to $\lambda_2=\frac{1}{2}$, and one to $\lambda_3=1$. To λ_1 belongs a space $\bar{S}_a$ of contravariant vectors $\bar{a}$, to λ_2 a space $\bar{S}_b$ of contravariant vectors $\bar{b}$, and to λ_3 a space $\bar{S}_l$ of vectors $\alpha\bar{l}$. We write $\overset{a}{H}{}_i^{\ j}=h^j_{\ i\alpha}\bar{a}^\alpha$, $\overset{b}{H}{}_i^{\ j}=h^j_{\ i\alpha}\bar{b}^\alpha$. The present paper studies the structure of these tensors $\overset{a}{H}$ and $\overset{b}{H}$.

D. J. Struik.

Moór, Arthur. Finslersche Räume mit der Grundfunktion $L=\dfrac{f}{g}$. Comment. Math. Helv. 24, 188–195 (1950).

The author classifies according to the quality of their invariants such two-dimensional Finsler spaces as have their

base function of the form $L=f/g$, f and g being both homogeneous polynomials of degree n, resp. $n-1$, in $\dot{x}$, $\dot{y}$. The main results are the following. In the case $n=2$, if the "Hauptskalar" J [Berwald, J. Reine Angew. Math. **156**, 191–210, 211–222 (1927)] is equal to zero, then (1) L must be of the form $A(x,y)\dot{x}+B(x,y)\dot{y}$; and if $J=\text{const.}$, then (2) $L=(C_0\dot{x}+C_1\dot{y})^2/(D_0\dot{x}+D_1\dot{y})$ and $J=-3/2\sqrt{2}$. In the case $n=3$, if $f=(C_0\dot{x}+C_1\dot{y})^3$ and $J=\text{const.}$, we get (1), (2), or (3) $J=-5/2^{\frac{3}{2}}3^{\frac{1}{2}}$; if $f=(A_0\dot{x}+A_1\dot{y})^2(C_0\dot{x}+C_1\dot{y})$ and $J=\text{const.}$, then (2) follows. If $L=A\dot{x}^n/B\dot{y}^{n-1}$, then $J=-(2n-1)/2[n(n-1)]^{\frac{1}{2}}$. In the last case the Finsler space defines a Minkowski geometry when its curvature scalar is equal to zero, that is, $\log A/B$ has the form $\alpha(x)+\beta(y)$.

A. Kawaguchi (Sapporo).

NUMERICAL AND GRAPHICAL METHODS

Carrus, Pierre A., and Treuenfels, Charlotte G. Tables of roots and incomplete integrals of associated Legendre functions of fractional orders. J. Math. Physics **29**, 282–299 (1951).

In connection with the scattering of electromagnetic waves from a conductor of conical shape one needs roots of the equations $f(n)=0$ and $g(n)=0$ where $f(n)=P_n{}^1(\cos\theta)$, $g(n)=(d/d\theta)P_n{}^1(\cos\theta)$ and $P_n{}^m$ is the associated Legendre function. The integral $I=\int_{x_0}^{0}[P_\nu{}^1(x)]^2dx$ is also needed. The present paper contains a discussion of the equations, expansions, and numerical tables of the first 50 zeros of $f(n)$ to 5S for $\theta=90°(5°)175°$, the first 51 zeros of $g(n)$ to 5S for $\theta=90°(5°)130°$, the integrals I for the first fifty zeros of $f(n)$ and $x=\cos\theta$ where $\theta=95°(5°)145°$, for the first fifty zeros of $g(n)$ and $x=\cos\theta$ where $\theta=95°(5°)130°$, and also for $n=0(.05)1$ and $x=1$. The tables are a joint effort of several staff members of the M.I.T. Computation Laboratory.

A. Erdélyi (Pasadena, Calif.).

Sadler, D. H. Maximum-interval tables. Math. Tables and Other Aids to Computation **4**, 129–132 (1950).

On the assumption that a given function will be approximated by a series of Chebyshev polynomials

$$C_n(t)=2\cos(n \arccos \tfrac{1}{2}t),$$

expressions are obtained for the minimum number of tabular entries required to provide a specified maximum absolute or relative error in the interpolated values.

T. N. E. Greville (Washington, D. C.).

Kommerell, Karl. Berechnung der trigonometrischen und zyklometrischen Funktionen durch Kettenwurzeln. Math.-Phys. Semesterber. **2**, 126–134 (1951).

Ballantine, J. P. Solution of quadratic equations and triangles by machine. Amer. Math. Monthly **58**, 92–98 (1951).

The author introduces a symbolism to describe the action of an ordinary desk calculator; for instance, $K(n)$ is the operation of entering n on the keyboard, $T(1)$ the operation of transferring the quantity in the keyboard to the accumulator and thus $T(m)$ describes multiplication by m. Specific solutions to the problems mentioned in the title are described using this symbolism. The reviewer believes that this operational symbolism could be effectively supplemented by a symbolism for the state of the machine.

F. J. Murray (New York, N. Y.).

Hartree, D. R. Automatische Rechenmaschinen. Z. Angew. Math. Mech. **31**, 1–12 (1951).

Expository lecture given in April, 1950 at a meeting of the Gesellschaft für angewandte Mathematik und Mechanik.

Booth, Andrew D. Design principles of all purpose digital computers. Acta Physica Austriaca **4**, 85–97 (1950).

This is an exposition of elementary principles. The author discusses the main organization of a machine into an arithmetic unit, a memory, and a control, the scale of notation, serial vs. parallel operation, the arithmetic unit, the round-off procedure, the control, the memory, the size of the memory and numbers, the input and output, and the code. Only single address systems are considered.

H. B. Curry (Brussels).

McCallum, D. M., and Smith, J. B. Mechanized reasoning. Logical computors and their design. Electronic Engrg. **23**, 126–133 (1951).

This paper describes in detail a small logical computing machine consisting of 17 relays and 1 stepping switch which was built by the authors at Messrs. Ferranti, Ltd. The paper includes extrapolations and speculations as to future developments along these lines.

R. Hamming (Murray Hill, N. J.).

Souriau, J.-M., et Bonnard, R. Théorie des erreurs en calcul matriciel. Recherche Aéronautique **1951**, no. 19, 41–48 (1951).

Let A be square matrix which is the erroneous representation of a correct matrix $A+B$. Then to a first order linearization, the corresponding correction for the inverse is $C=A^{-1}BA^{-1}$. The author shows how the size of terms of C can be estimated in terms of the bound $|B|$ of B and the norm of A^{-1}. (The norm is the square root of the sum of the squares of the elements of a matrix.) The elements of C are then discussed from a probability point of view and it is shown the expected value of $c_{i,j}\cdot c_{k,l}$ is less in absolute value than $K\cdot|A^{-1}|^2$, where K is a constant which depends on B and if the elements of B are independent with mean square α, $K=\alpha$. The discussion is given in terms of certain tensors and biquadratic forms, for instance $\Phi_{j,k}$, the expected value of $\sum_{i,l} b_{i,j}\bar{b}_{l,k}x_i\bar{y}_l$. The authors also give a discussion of the characteristic roots of A in an analogous manner. The "smallness" of B is taken into account by supposing that the correct matrix can be written in the form $A+\epsilon B$ and that the correct characteristic value and the correct right and left characteristic vectors can be written in the form $\lambda+\epsilon\lambda$, $X+\epsilon X$, and $Y+\epsilon Y$. A comparison of the terms linear in ϵ in the characteristic equation yields $\lambda=(\tilde{Y}BX)/YX$. However, the reviewer would like to point out that the assumption concerning the forms for the characteristic values and vectors does not hold in the case in which A has equal roots and nondiagonal Jordan normal form. The reviewer also believes that error estimates for the characteristic values should be sensitive to the possibility of nearly coincident roots.

F. J. Murray (New York, N. Y.).

Parodi, Maurice. Sur des familles de matrices auxquelles est applicable une méthode d'itération. C. R. Acad. Sci. Paris **232**, 1053–1054 (1951).

Plunkett [Quart. Appl. Math. **7**, 419–421 (1950); these Rev. **11**, 464] showed that (*) $u=u_0+Qu$, where u and u_0 are column vectors and Q is a real square matrix, can be solved by iteration, $v_{n+1}=\theta v_n+(1-\theta)[u_0+Qv_n]$ provided (**) the matrix $\theta I+(1-\theta)Q$ has all characteristic roots less than unity in absolute value. The author considers two cases where θ may be found so that (**) holds. Denote by $T=\max[|a_{ii}|/(1+|a_{ii}|)]$ where $Q=(a_{ij})$ and by $s_i=\sum|a_{ij}|$ where the sum is for $j\neq i$. Case I: $a_{ii}>1$, $s_i<a_{ii}-1$; then (**) holds for $T>\theta>1$. Case II: $a_{ii}<0$, $s_j<1+|a_{ii}|$; then (**) holds for $T<\theta<1$.

M. A. Woodbury (Princeton, N. J.).

Andree, R. V. Computation of the inverse of a matrix. Amer. Math. Monthly **58**, 87–92 (1951).

If A is a nonsingular matrix and $PAQ=I$ then $A^{-1}=QP$. A method for the computation of the inverse based on this observation is developed. The author seems unaware of the extensive literature on this topic. The paper of L. Guttman [Ann. Math. Statistics **17**, 336–343 (1946); these Rev. **8**, 171] presents an allied procedure; H. Hotelling [ibid. **14**, 1–34, 440–441 (1943); these Rev. **4**, 202; **5**, 245] gives a bibliography and I. Schur [J. Reine Angew. Math. **147**, 205–232 (1917), especially p. 217] seems to provide the earliest relevant reference. *M. A. Woodbury.*

Bartlett, M. S. An inverse matrix adjustment arising in discriminant analysis. Ann. Math. Statistics **22**, 107–111 (1951).

Let (*) $B=A+uv'$ be a square matrix where u and v are column matrices so that uv' is a matrix of rank one. Then (**) $B^{-1}=A^{-1}-A^{-1}u(1+v'A^{-1}u)^{-1}v'A^{-1}$ which is in essentially the same form as (*) since $v'A^{-1}u$ is a scalar. This result includes that of Sherman and Morrison [same Ann. **21**, 124–127 (1950); **20**, 621 (1949); these Rev. **11**, 693]. This result is closely related to those of the reviewer [The Stability of Output-Input Matrices, Chicago, Ill., 1949; Statistical Research Group, Memo. Rep. no. **42**, Princeton University, Princeton, N. J., 1950; these Rev. **11**, 307; **12**, 361], where it is shown that (**) holds in suitably generalized form when u and v are rectangular matrices. Two applications to discriminatory analysis are made.

M. A. Woodbury (Princeton, N. J.).

Šreĭder, Yu. A. The solution of systems of linear consistent algebraic equations. Doklady Akad. Nauk SSSR (N.S.) **76**, 651–654 (1951). (Russian)

The author proposes a direct method for solving a system $Ax=b$ which, he claims, will apply equally well to all "well-conditioned" systems, whether or not A is positive definite. Successively orthogonalizing the rows of A by the Gram-Schmidt process (the author calls it the Jacobi process), the author gets the transformed system $BAx=Bb$. Since $(BA)^{-1}=(BA)^*D$, D diagonal, one has $x=(BA)^*DBb$. The orthogonalization, the solution, and a more precise operation count may be found in Bargmann, Montgomery, and von Neumann, Solution of Linear Systems of High Order [Institute for Advanced Study, Princeton, N. J., 1946].

G. E. Forsythe (Los Angeles, Calif.).

Soulé-Nan, Geneviève, et Couffignal, Louis. Recherche des racines réelles et positives d'une équation transcendante. Revue Sci. **88**, 14–16 (1950).

Solution of a transcendental equation treated, by other means, by Eckart and Kahan [Revue Sci. **86**, 723–726 (1948); these Rev. **11**, 403]. *Z. Nehari.*

Slibar, A. Freie und erzwungene nichtlineare Schwingungen von Mehrmassensystemen. Österreich. Ing.-Arch. **4**, 398–408 (1950).

The systems studied are chains of masses connected together by nonlinear springs. It is proposed to solve the corresponding differential equations numerically by replacing derivatives by differences in the usual way. The work is assisted by a simple geometric construction. No analysis of errors is given. No numerical computations are carried through to illustrate the amount of labor involved. As with other similar numerical methods applied to nonlinear differential equations, this method is primarily useful when the systems contain no unspecified parameters, when the asymptotic form of the solution is not of interest, and when the investigator has access to sufficiently elaborate computational facilities.

E. Pinney (Berkeley, Calif.).

Hopkin, H. R. Routine computing methods for stability and response investigations on linear systems. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2392** (**10014**), 50 pp. (1950).

A number of computing methods (some old, some new) for solving linear differential equations with constant coefficients are collected together. The major portion of the report deals with the solution of polynomial equations, and general methods are given for finding real and complex roots; fourth and sixth order equations are given special attention. Other sections describe methods for obtaining stability diagrams of two types, and for reducing the labour in response calculations when the stability equation has complex roots. Many worked examples illustrate the methods.

Author's summary.

de Vogelaere, René. Équation de Hill et problème de Störmer. Canadian J. Math. **2**, 440–456 (1950).

The author considers two methods for the solution of Hill's equation: (1) by numerical integration of the equation; (2) by the evaluation of an infinite determinant. According to the first method he gives an expression for the characteristic exponent in terms of two special independent solutions of the problem. This expression is examined in the case of periodic solutions. By way of examples the equatorial orbits of Störmer's problem are considered in which an electrified particle moves in the field of a dipole. The second method is considered briefly and is shown to be capable of improving the limiting orbits obtained by the first method in certain simple cases. Considering Störmer's problem, the author first states some known results, then he calculates an equatorial orbit numerically as well as the corresponding characteristic exponents. The approximate and the exact results for some equatorial orbits are compared numerically. It is shown that not only the characteristic component, but also the complete solution may be calculated numerically.

M. J. O. Strutt (Zurich).

Murray, F. J. Planning and error considerations for the numerical solution of a system of differential equations on a sequence calculator. Math. Tables and Other Aids to Computation 4, 133–144 (1950).

This paper is concerned with the solution on the IBM selected sequence calculator of a specific system of fourteen ordinary differential equations $z_i' = f_i(t; z_1, \cdots, z_n)$, $i = 1, 2, \cdots, n = 14$, where the functions f_i, apparently, involve certain nonanalyticities because of the presence of the absolute-value function. The author discusses the general principles back of certain decisions involving computational technique which were made in planning the above computation. The questions discussed include the choice of method (including such practical details as the choice of interval and the simplifications of the step-by-step procedure), checks on the validity of the assumptions, the effect of the nonanalyticities, and an elaborate error-analysis. The last is similar in spirit to that of Rademacher [Proceedings of a Symposium on Large-Scale Digital Calculating Machinery, Annals of the Harvard Computation Lab., v. 16, pp. 176–187, Harvard University Press, Cambridge, Mass. 1948; these Rev. 9, 468], but differs considerably in the details. *H. B. Curry* (Brussels).

Flanders, Donald A., and Shortley, George. Numerical determination of fundamental modes. J. Appl. Phys. 21, 1326–1332 (1950).

Die Verfasser behandeln zunächst das Eigenwertproblem bei mehrfach durchlöcherten Membranen, also $\Delta u + \alpha u = 0$, wobei $u = 0$ längs der Berandung gelten soll. Der Differentialoperator wird durch den analogen Differenzenausdruck ersetzt. Sei ω jener lineare Operator, der einem Gitterpunkt P eines engmaschigen, quadratischen Netzes den Mittelwert aus den vier Nachbarpunkten von P zuordnet; dann kann das analoge algebraische Eigenwertproblem in der Form $\omega u_\alpha = \lambda u_\alpha$ geschrieben werden, wobei u_α die Werte in den einzelnen Gitterpunkten bedeuten. Diese u_α können als Komponenten eines Vektors aufgefasst werden, deren absolute Beträge durch die Operation ω verkleinert werden. Hieraus folgt $-1 < \lambda < 1$. Sei nun $v = \sum_1^n c_i u_i$ ein beliebiger, den Randbedingungen entsprechender Vektor, wobei u_i die Eigenvektoren darstellen. Sei ferner $P(\omega)$ ein linearer Operator, der aus ω durch Polynombildung hervorgegangen ist. Dann wird $P(\omega) \cdot v = \sum_n P(\lambda_n) c_n u_n$ wobei die Faktoren $P(\lambda_i)$ den Wert bedeuten, den das Polynom P für den Eigenwert λ_i annimmt. Es kommt offenbar darauf an, das Polynom P so zu wählen, dass etwa der erste und der zweite Faktor verhältnismässig gross sind, während alle übrigen Faktoren sehr klein sein sollen. Denn, stellt v einen Vektor dar, der mit geringer Vernachlässigung als lineare Kombination der ersten beiden Eigenvektoren angesehen werden kann, so kann man durch

$$\begin{aligned} (v \cdot v) &= C_1^2 + C_2^2, \\ (v \cdot \omega v) &= \lambda_1 C_1^2 + \lambda_2 C_2^2, \\ (v \cdot \omega^2 v) &= \lambda_1^2 C_1^2 + \lambda_2^2 C_2^2, \\ (\omega v \cdot \omega^2 v) &= \lambda_1^3 C_1^2 + \lambda_2^3 C_2^2, \end{aligned} \qquad (v = C_1 u_1 + C_2 u_2)$$

C_1, C_2, λ_1 und λ_2 ermitteln. Die Verfasser verdanken den Herrn Tukey und Grosch die Anregung, als Polynom P ein solches zu verwenden, das aus den Tschebyscheff-Polynomen durch eine geeignete lineare Transformation der unabhängigen Variablen hervorgeht und so die bekannte Minimaleigenschaft der Tschebyscheff-Polynome $T_n(x)$ für das Intervall $(-1, 1)$ auf ein geeignet gewähltes Intervall überträgt. Im letzten Kapitel werden Verallgemeinerungen besprochen, wobei auch die Behandlung von nichtsymmetrischen Operatoren erörtert wird. Den Anlass zu dieser Arbeit bildeten Anwendungen auf zwei- und dreidimensionale Probleme der Diffusionstheorie. *P. Funk.*

Maslov, P. G. On the method of determining the vibrations of polyatomic molecules. The method of combined steepest descent. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 20, 609–618 (1950). (Russian)

In order to determine from observed vibrational frequencies of a group of molecules of similar structure the coefficients in the quadratic expression for the potential energy, it is necessary to use a procedure of successive trials and adjustments. The feasibility of such a calculation depends on the availability of an efficient method for solving secular equations. The author presents a method and illustrates it by a numerical example. *W. H. Furry.*

✱de Beauclair, W. Verfahren und Geräte zur mehrdimensionalen Fouriersynthese. Akademie-Verlag, Berlin, 1949. viii+71 pp. (1 plate).

This booklet deals with mechanical aids and computational schemes facilitating Fourier synthesis as practised in crystallographic X-ray analysis. It is arranged in three parts dealing respectively with one-, two-, and three-dimensional synthesis. In each part the computational procedures are divided into those synthesising Fourier series as continuous curves and those which only provide the series at discrete grid values. The latter are mainly digital and utilise as mechanical aids trigonometric tables, strips and stencils, calculating machines, accounting machines, and punched card equipment, the former are mainly based on analogue principles of a mechanical, electrical, and optical nature. The author recomends for particular attention a method of electrical voltage addition of a battery of varying A/C voltages with adjustable frequencies, amplitudes and phases. A planned construction of such an instrument is described in detail although its completion was prevented by the war, whilst many existing and working aids are only cursorily described. In particular, the method of using prepared multiplication tables in the form of strips [e.g. the Beevers-Lipson strips] are somewhat lightly dismissed. The punched card methods for two- (or three-) dimensional synthesis are based on a German Hollerith multiplying punch and work on the principle of "group-multiplying" x-arrays and y-arrays by constant trigonometrical multipliers. In general, the emphasis is too much on the analogue instruments at the expense of the digital procedures. *H. O. Hartley.*

Goflin, A. L. An electrical apparatus for harmonic analysis and synthesis. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1950, 1125–1136 (1950). (Russian)

The present apparatus, operating on essentially the same principle as that built by the author for Tolstov [same journal 1946, 389–400; these Rev. 8, 287], is offered as a marked improvement since the only connections that need be made in using it are those corresponding to the problem in hand.

To realize the Bessel formulas $B_k = m^{-1} \sum_{i=1}^{2m} y_i \sin(i\pi k/m)$, $k = 1, \cdots, m-1$ (and those for A_0, A_m, and A_k) there is a transformer T_1 with $2m$ secondary coils, each tapped and controlled by plugwires so that potentials proportional to the y_i are available. For each i there is a transformer T_{2i} whose primary is across the corresponding secondary coil of T_1 and which has $2m$ secondary coils. One secondary coil of each T_{2i} has w_0 turns; these $2m$ coils are connected in series

to determine A_0. And $m-1$ of the remaining coils on T_{2i} have w_{hk} turns, where $w_{hk}=w_0 \sin (i\pi k/m)$, $k=1, \cdots, m-1$. These are connected in series, in sets, to determine the B_k. The remaining secondary coils are similarly determined and arranged for the A_k which together with the B_k are measured using a balance circuit. The author recommends a second apparatus arranged for synthesis to check the result of analysis, and conversely. The device, which has been built for $m=12$, permits entering the y_i to two significant figures. The examples of analysis and synthesis that are reported show errors that do not exceed 1.5%. The reviewer notes that m^{-1} is used throughout as a coefficient in the Bessel formula for A_m instead of the usual $(2m)^{-1}$.

R. Church (Annapolis, Md.).

Redheffer, R. M. Calculating machine for Fourier transforms and related expressions. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. no. 13, ii+41 pp. (1946).

Turetsky, R. The least squares solution for a set of complex linear equations. Quart. Appl. Math. 9, 108–110 (1951).

The author gives a method for finding that complex vector z ($n\times 1$ matrix) which is the least squares solution for the set of m complex linear equations $Az=w$, where A is an $m\times n$ ($m\geqq n$) matrix of rank n whose elements are prescribed complex quantities, while w is a vector (of observations) represented by an $m\times 1$ matrix. The vector z minimizes the sum of the squares of the absolute values of the components of the vector $Az-w$. The results of this paper constitute a generalization of the well-known least squares solution when the matrix A has real coefficients.

B. Epstein (Detroit, Mich.).

Guest, P. G. Estimation of the errors of the least-squares polynomial coefficients. Australian J. Sci. Research. Ser. A. 3, 364–375 (1950).

Continuation of an earlier paper on this subject [same vol., 173–182 (1950); these Rev. 12, 513]. *B. Epstein.*

Ansermet, A. De la pratique des calculs de compensation. Bull. Tech. Suisse Romande 77, 31–35 (1951).

Numerical illustrations of the techniques developed in an earlier paper by W. K. Bachmann [same Bull. 76, 74–77 (1950); these Rev. 12, 362]. *B. Epstein.*

Banachiewicz, T. Sur l'ajustement des observations dans le cas où les équations ne sont pas linéaires. Bull. Int. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1950, 113–114 (1950).

Säuberli, Rud. Graphische Ausgleichung. Schweiz. Z. Vermessg. Kulturtech. 49, 68–75 (1951).

Marussi, Antonio. Les principes de la géodésie intrinsèque. Bull. Géodésique N.S. 1951, 68–76 (1951).

ASTRONOMY

Krat, V. The rotation of the sun. Uspehi Astronom. Nauk 3, 129–145 (1947). (Russian)

The author briefly summarizes our present observational knowledge of the variation of the period of rotation of the sun with latitude and proceeds to analyze the various theories which have been proposed to explain the phenomenon. The older theories of Faye and Jeans, which assumed barotropic rotation, are rejected as unsatisfactory. A better approach, according to the author, was followed by Bjerknes, Rosseland, Klauder, and others, who adopted a barocline model, in which there is no acceleration potential. The most comprehensive attacks on the problem were launched by von Zeipel and Eddington, who applied to stellar rotation the theory of energy transfer which had been successfully used in stellar atmospheres. The author concludes that the problem of the rotation of the sun is not an independent problem, but must be considered as a particular case in the hydrodynamics of gas spheres. The equatorial acceleration observed on the sun requires no auxiliary hypotheses for its explanation. It is a simple consequence of the generation of energy in the interior of stars in general. If the stars did not generate energy, their rotation would proceed with constant angular velocity throughout. *L. Jacchia.*

Ramsey, W. H. The instability of small planetary cores. I. Monthly Not. Roy. Astr. Soc. 110, 325–338 (1950).

The radius of a spherical configuration of matter in hydrostatic equilibrium is not uniquely defined by its mass unless certain restrictions are imposed upon the equation of state. An algebraical examination is given of a case in which matter is assumed to be incompressible below and above a certain critical pressure, representing a phase transition with a discontinuous rise in density. Then the radius is unambiguously specified by the mass, both for small and large masses. But over an intermediate range, there are three different solutions to the hydrostatic problem: one without a core, called A, and two with cores, called B and C. The order A, B, C, is the order of increasing central pressures and also of decreasing total radius. The configurations A and C prove stable, and B is unstable; as is to be expected by Poincaré's principle of exchange of stabilities. The author suggests that the transition from an unstable to a stable configuration has occurred in an unidentified member of the solar system, resulting in fragmentation of the original body, and that this mechanism may have been responsible for the meteorites, and possibly also for the asteroids.

R. Wildt (New Haven, Conn.).

Lighthill, M. J. On the instability of small planetary cores. II. Monthly Not. Roy. Astr. Soc. 110, 339–342 (1950).

Results of part I [see the preceding review] are shown to hold even if the assumption of incompressibility is relaxed, provided that the density, at the critical pressure, rises by a factor greater than $\frac{3}{2}$. *R. Wildt.*

Gurevič, L. È., and Levin, B. Yu. On the formation of double stars. Akad. Nauk SSSR. Astr. Žurnal 27, 273–284 (1950). (Russian)

It is known that gravitational systems cannot be in a state of complete statistical equilibrium. Ambarzumian, however, has shown that the time of dissipation of gravitational systems considerably exceeds the time of relaxation, permitting a condition of quasi-equilibrium. This paper deals with the formation and dissolution of binary systems in such conditions of quasi-equilibrium in the galaxy and in open clusters. The author reaches the following conclusions:

(1) The time required by triple stellar approaches to form the equilibrium number of double systems is comparable with the time it takes to dissolve such systems through accidental encounters with single stars. (2) The evolution of a binary system due to stellar approaches depends, statistically, on the ratio of its orbital energy to the energy of motion of the passing star. Close binaries tend to become closer, open systems looser. This is in contradiction with the conclusion of Jeans, who found that open systems become closer through stellar encounters. (3) The time of formation of open pairs is less than the time of relaxation; for close pairs it exceeds the time of relaxation and increases as $a^{-\frac{1}{2}}$. (4) The energy of dissociation of stellar pairs, compared with the mutual gravitational interaction, leads to the concept of a "stellar gas", in which the equilibrium condition is reached when the "gas" condenses into one close multiple system. The results on the time of formation of close systems, reported in the previous paragraph, precludes the establishment of a condition of equilibrium. (5) To explain the abundance of double, and especially open double systems, in the vicinity of the sun, which is not justified by near-equilibrium conditions compatible with the known stellar density in our region of the galaxy, the author advances the hypothesis that these double systems were formed in stellar clusters, from which they subsequently "evaporated." *L. Jacchia* (Cambridge, Mass.).

Mineur, Henri. Recherches théoriques sur les accélérations stellaires. Ann. Astrophysique 13, 219–242 (1950).

In part I purely kinematic problems are considered, in particular, the perspective acceleration in the spherical coordinates of a star having a uniform motion in space relative to the Sun. Part II deals with the acceleration by the galactic system as a whole upon an individual star. Part III considers the acceleration of the sun produced by the outer planets and by a hypothetical nearby star. The theoretical results are compared with the accuracy attainable with precise astrometry. Barnard's star is the most favorable case for which a measured acceleration, in good agreement with calculations, is available. The acceleration produced by Jupiter's attraction on the sun is found to be just measurable in the nearest stars. The remaining effects are found to be too small to yield measurable effects at present, but will become significant with the lapse of time. *D. Brouwer.*

Tuominen, Jaakko. A note on the theory of stellar dynamics. I. Nederl. Akad. Wetensch., Proc. **53**, 1049–1055 (1950).

Keplerian orbits in the field of a central mass M with eccentricities less than a given $e_0=\sigma_0(2-\sigma_0)$, with the longitude of the apcentron taking all values from 0 to π with equal probability, and uniformly filling a plane with a surface density $\rho=-N\log(1-\sigma_0)=N\sigma_0(1+\frac{1}{2}\sigma_0+\frac{1}{3}\sigma_0^2+\cdots)$, is considered. It is shown that for this system the mean radial velocity ($\bar{R}$) is zero while the mean transverse velocity ($\bar{T}$) is $(GM/r)^{\frac{1}{2}}(1-\frac{1}{12}\sigma_0^2+\cdots)$; also that if $U=R$ and $V=T-\bar{T}$ then $\overline{U^2}=2(GM/r)\sigma_0^2/3$, $\overline{V^2}=(GM/r)\sigma_0^2/6$ and $\overline{UV}=0$; these latter expressions for $\overline{U^2}$ and $\overline{V^2}$ are correct to $O(\sigma_0^2)$. *S. Chandrasekhar* (Williams Bay, Wis.).

Tuominen, Jaakko. A note on the theory of stellar dynamics. II. Nederl. Akad. Wetensch., Proc. **53**, 1211–1216 (1950).

In this paper Keplerian orbits in the field of a central mass M with eccentricities less than a given $e_0=\sigma_0(2-\sigma_0)$ are considered; in contrast to the paper reviewed above it is assumed here that the longitude of the apcentron of the orbit is either zero or π with equal probability $\frac{1}{2}$. With a surface density of $\frac{1}{2}N\log(1-\sigma_0)$ in each of the subsystems, it is shown that for this "most oval system"

$$\bar{R}=-\frac{\sigma_0^2}{3}\left(\frac{GM}{r}\right)^{\frac{1}{2}}\sin 2\vartheta;\quad \bar{T}=\left(\frac{GM}{r}\right)^{\frac{1}{2}}\left(1-\frac{1}{6}\sigma_0^2\cos^2\vartheta\right),$$

$$\overline{U^2}=\overline{(R-\bar{R})^2}=\frac{4\sigma_0^2}{3}\frac{GM}{r}\sin^2\vartheta;$$

$$\overline{V^2}=\overline{(T-\bar{T})^2}=\frac{\sigma_0^2}{3}\frac{GM}{r}\cos^2\vartheta \quad\text{and}\quad \overline{UV}=\frac{\sigma_0^2}{3}\frac{GM}{r}\sin^2\vartheta.$$

S. Chandrasekhar (Williams Bay, Wis.).

Stehle, P. Dynamics of star streaming. II. Astrophys. J. **112**, 299–306 (1950).

In this paper the author extends his earlier work on the reviewer's treatment of stellar dynamics [same J. **110**, 250–260 (1949); these Rev. **11**, 466]. The potential energy of the system is assumed to be subject to two first order differential equations and it is shown that as a consequence only systems with plane, cylindrical, or spherical symmetry are possible. The results are applied to a system with a central mass when the potential has spherical symmetry.

S. Chandrasekhar (Williams Bay, Wis.).

McVittie, G. C. The expansion of an interstellar gas-cloud into a vacuum. Monthly Not. Roy. Astr. Soc. **110**, 224–237 (1950).

This paper is concerned with the one-dimensional problem of the expansion of a gas cloud into a vacuum. The cloud extends to infinity on one side of its infinite plane face; on the other side of the face there is a vacuum into which the cloud is advancing. It is assumed that the pressure and the density satisfy the adiabatic law $p=\kappa\rho^k$ (κ and k constants) and that the effects of viscosity and heat conduction can be neglected. In terms of the Riemann variables $r=\frac{1}{2}u+c/(k-1)$ and $s=-\frac{1}{2}u+c/(k-1)$, where $c=(kp/\rho)^{\frac{1}{2}}$ denotes the local velocity of sound, the equations of motion and continuity are

$$\partial r/\partial t+(\alpha r+\beta s)\partial r/\partial x=0 \tag{1}$$

and

$$\partial s/\partial t-(\beta r+\alpha s)\partial s/\partial x=0,$$

where $\alpha=\frac{1}{2}(k+1)$ and $\beta=\frac{1}{2}(k-3)$. If neither $\partial r/\partial x$ nor $\partial s/\partial x$ are zero, r and s can be chosen as the independent variables. It can then be shown that $dw(r,s)=Xdr-Yds$, where $X=x-(\alpha r+\beta s)t$ and $Y=x+(\beta r+\alpha s)t$ is a perfect differential and that w satisfies the differential equation

$$\frac{\partial^2 w}{\partial r\partial s}+\frac{n}{r+s}\left(\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}\right), \tag{2}$$

where $n=(3-k)/2(k-1)$. The author discusses the boundary conditions of the problem very carefully and shows that the front B_1 (at x_1) advancing into the vacuum must move with the velocity of the gas u_1 at x_1. The gas velocity v_2 at the surface B_2 (at x_2) progressing into the gas at rest must be zero and for the solution of the equations this implies different things, depending upon whether the gas cloud is initially homogeneous or not. In the homogeneous case, it appears consistent with the problem to assert that $s=\text{constant}=-c_0/(k-1)$ where c_0 is the velocity of sound in the undisturbed medium and that the front B_2 progresses

into the gas at rest with a velocity equal to the local velocity of sound c_0. If the gas cloud is initially nonhomogeneous, the author supposes that the velocity of sound $-c_0(x)$ can be assigned arbitrarily. The resulting boundary conditions at x_2 are:

$$2\left(\frac{\partial r}{\partial x}\frac{\partial s}{\partial x}\right)_{x_2}=-\frac{1}{k-1}\left\{\frac{\partial c_0}{\partial x}\left(\frac{\partial r}{\partial x}+\frac{\partial s}{\partial x}\right)\right\}_{x_2} \tag{3}$$

and

$$v_2=-\left\{c_0\left(\frac{\partial r}{\partial x}+\frac{\partial s}{\partial x}\right)\Big/\left(\frac{\partial r}{\partial x}-\frac{\partial s}{\partial x}\right)\right\}_{x_2}. \tag{4}$$

These boundary conditions can be translated in terms of the derivatives of w for solving equation (2).

The solution for the case of the homogeneous cloud is carried out with equations (1). Since

$$s=\text{constant}=-c_0/(k-1),$$

the only equation to be considered is that for r and the general solution for this is found to be $x-t[\alpha r-\beta c_0/(k-1)]=f(r)$, where $f(r)$ is an arbitrary function of r. The solution can now be completed and the extent of the cloud in motion is given by $v_1=2c_0/(k-1)$, $x_1-2c_0t_1/(k-1)=f(c_0/(k-1))$ and $v_2=-c_0$, and $x_2-v_2t_2=f(-c_0/(k-1))$. For the case $f\equiv 0$ the solution reduces to that given by Burgers [Nederl. Akad. Wetensch., Proc. **49**, 589–607 (1946)]. However, the author considers Burgers' solution as "nonphysical" since the front advancing into the vacuum moves with the high speed $2c_0/(k-1)$ $(=3c_0$ for $k=5/3)$. He therefore considers the case of a nonhomogeneous cloud and restricts himself to the case $k=5/3$ and $n=1$; for this case the general solution of equation (2) can be written in the form (5) $w(r,s)=[R(r)+S(s)]/(r+s)$, where $R(r)$ and $S(s)$ are two arbitrary functions of the argument. The boundary condition at B_1 namely $r+s=0$ (corresponding to ρ and $c=0$) can be satisfied if $S(s)=-R(-s)$. The position of the advancing front B_1 is given by $x_1-v_1t_1=\frac{1}{2}R''(\frac{1}{2}v_1)$ where $v_1=u_1$ is the velocity of the front. The receding edge B_2 is specified by

$$x_2=(R_+'-R_-')/4r_2,\quad t_2=3\{-r_2(R_+'+R_-')+(R_+-R_-)\}/8r_2^3$$

and

$$v_2=2r_2^2\{R_+'-R_-'-r_2(R_+''+R_-'')\}/\\ 8\{3(R_+-R_-)-3r_2(R_+'+R_-')+r_2^2(R_+''-R_-'')\},$$

where $R(r_2)=R_+$, $R'(r_2)=R_+'$, $R(-r_2)=R_-$, etc. Finally the author shows that the boundary condition (3) can be integrated to give

$$R(r_2)+R(-r_2)=2\left\{C+x_2r_2^2-\int r_2^2dx_2\right\}, \tag{6}$$

where C is a constant of integration and $r_2=-\frac{3}{2}c_0(x_2)$ is a known function of r_2. The author illustrates his solution for the nonhomogeneous case by considering the case $c_0^2=a^2-\frac{4}{9}(\frac{3}{2}Aa/(-x_2))^{\frac{1}{2}}$ $(x<0)$ where A and a are certain positive constants. For this case $R(r)=-A/(\frac{3}{2}a-r)$ and the entire solution can be explicitly worked out. If it is assumed that when the expansion began B_1 and B_2 coincided, then the face of the cloud is at the absolute zero of temperature and the motion starts with B_2 moving off to the left at the speed a while B_1 remains momentarily at rest. At B_1, $\partial r/\partial x$ and $\partial s/\partial x$ are both infinite and the high speed, $2c_0/(k-1)$, of the advancing face found by Burgers is attained in this example only as a limit after the expansion has proceeded for an infinite time. *S. Chandrasekhar.*

Copson, E. T. The expansion of a gas cloud into a vacuum. Monthly Not. Roy. Astr. Soc. **110**, 238–246 (1950).

This paper treats the same problem considered by McVittie [see the preceding review]. The author remarks that the assumption of nonhomogeneity by McVittie, namely, that the local velocity of sound $c_0(x)$ can be specified arbitrarily before the expansion starts, is inadmissible since the initial acceleration $[=-2c_0(x)c_0'(x)/(k-1)]$ will be nonzero and the cloud will start moving contrary to hypothesis. The author therefore considers that the cloud is initially at rest and is homogeneous except for a boundary layer of arbitrary thickness in which $c_0'(x)$ is negative. Motion starts in the nonhomogeneous layer and spreads into the cloud. The author restricts himself to the case $k=5/3$ and works out the complete solution in terms of the general solution of the equation for w discovered by McVittie [see eq. (5) of the preceding review]. The particular boundary conditions the author adopts are: $r=s=b$ $(x<0)$ and $r=s=r_0(x)$ $(0\leqq x\leqq h)$, where $r_0(x)$ is a given function and $r_0(0)=b$, and $r_0(h)=0$, and $b=\frac{3}{2}a$, a being the velocity of sound in the undisturbed medium. It is shown that under these circumstances the initial boundary of the nonhomogeneous layer will split into two boundaries and that the state at a slightly later time will be as follows: for $x<x_3(t)$, $r=s=b$; for $x_3(t)<x<x_2(t)$, $r=b$ and $s=s'(x,t)$ and for $x_2(t)<x<x_1(t)$, $r=r(x,t)$ and $s=s(x,t)$. The positions of the two boundaries x_2 and x_3 are a priori unknown; it must be required that $r(x_2,t)=b$, $s(x_2,t)=s'(x_2,t)$ and $s'(x_3,t)=b$. The author solves this problem and shows that during this stage of the motion the front x_2 advances with a speed greater than that of the front x_1 and must therefore catch up with it. During the second stage of the motion which then follows the Riemann variable r is everywhere constant. The resulting state of motion during the second stage is similar to that discovered by Burgers [Nederl. Akad. Wetensch., Proc. **49**, 589–607 (1946)]; in particular the front x_1 now advances with a speed equal to $3c_0$. The author suggests that the difficulty in Burgers' solution pointed out by McVittie regarding the discontinuity in the initial state can be resolved by regarding Burgers' solution as the limit of the one found by the author when the thickness of the initial layer of nonhomogenity is made to vanish. *S. Chandrasekhar* (Williams Bay, Wis.).

Carrus, Pierre A., Fox, Phyllis A., Haas, Felix, and Kopal, Zdeněk. Propagation of shock waves in the generalized Roche model. Astrophys. J. **113**, 193–209 (1951).

The stellar model considered is spherically symmetrical and consists of a heavy core and of a mantle whose own gravitational field is neglected and whose density is proportional to the inverse square of the distance from the centre. A shock is supposed to be produced by the core suddenly expanding. It is found that a similarity solution exists in which the Mach number of the shock takes a constant (arbitrary) value, owing to spherical attenuation being cancelled out by propagation into a medium of lower density. The solution is an exact solution of the Euler equations of motion, including the effect of variable gravity. The quantity p/ρ^γ is supposed constant for any fluid element, but not necessarily uniform. The form of the velocity, for example, is $rt^{-1}U(r^{-3}t^2)$. For $\gamma>\frac{4}{3}$ a contact discontinuity appears in the solution, corresponding to the boundary between core and mantle. The similarity solution has been computed for $\gamma=\frac{5}{3}$ and $\gamma=\frac{4}{3}$, each with a wide range of values of the shock Mach number. The results are set out in great detail in the paper, and they include data on the

matter ejected when the shock is reflected from the outer boundary of the mantle. For infinite shock Mach number there is a very simple exact solution if $\gamma=\frac{5}{3}$, in which $U=\frac{1}{2}$. This is shown to be one of a series of particularly simple exact solutions for infinite shock Mach number and various density fields and values of γ, including the previously known case of uniform density and $\gamma=7$. The authors have made an even more striking advance in discovering exact solutions for arbitrary shock Mach numbers, in the special case when the density varies as r^{-2}, with or without a gravitational field. *M. J. Lighthill* (Manchester).

Sobolev, V. V. The illumination of stellar envelopes in the absence of radiative equilibrium. Akad. Nauk SSSR. Astr. Žurnal **27**, 81–88 (1950). (Russian)

Radiative equilibrium is usually assumed in problems concerning stellar envelopes. In certain cases, however, this assumption is hardly justifiable. In a previous paper [Vestnik Leningrad Univ. No. 10, 1948], the author briefly sketched the outlines of a theory involving a non-stationary field of radiation. Here he proceeds to work out in detail a theory of the spectral changes in the course of a Nova explosion, under the assumption that the Nova envelope radiates like a planetary nebula, i.e., as a result of photo-ionization and recombinations. The temperature of the star is supposed to rise suddenly; ultraviolet radiation produces ionization and, in turn, increase in brightness, until ionization equilibrium corresponding to the increased temperature is established. Two cases are considered: In the first the optical depth of the stellar envelope is small; in the second it is large. In the first case the differential equation which governs the change of ionization with time is easily integrated and the time of relaxation can be computed without further assumptions. The solution is more complicated in the case of a large optical depth; here the integration of the ionization equation is performed by changing over to a new independent variable $u=tn_e\sum_2^\infty C_i$, where t is the time, n_e the free-electron density, and C_i the coefficient of electron capture for the ith level. The relation between n_1/n (ratio of neutral atoms to all atoms) and τ_0 (optical distance of the given point when $t=0$) for given values of u consists of a family of curves, each of which shows a very sharp rise at a different value of τ_0. This indicates that the boundary between the ionized and the non-ionized region in the envelope is quite sharp. Expressions are given for the time of relaxation and for the energy radiated in a spectral line of given frequency. Spectroscopic observations of Nova Herculis 1934 by C. Payne-Gaposchkin and F. L. Whipple [Harvard College Observatory, Circular no. **433** (1939)] are used by the author for some computations on the basis of his theory. The time of relaxation turns out to be 1.3×10^6 seconds and, assuming an electron temperature of $10\,000°$, $n_e=3\times10^6$ electrons per cm³. The energy radiated by the envelope in H_γ is 2.7×10^{33} erg/sec, the corresponding number of ions is $N^+=1.4\times10^{52}$ and, finally, the mass of the star $M=2.3\times10^{28}$ grams.

L. Jacchia (Cambridge, Mass.).

Chandrasekhar, S., and Münch, G. The theory of the fluctuations in brightness of the Milky Way. I. Astrophys. J. **112**, 380–392 (1950).

The authors consider the problem of the fluctuation in brightness of the Milky Way by a statistical approach, and aim at determining the probability distribution $g(I, L)$ of the observed brightness I of a stellar system in which stars and absorbing interstellar clouds are interspersed at random. Let this system extend to a linear distance L in the direction of the line of sight, and let each cloud reduce the intensity of light of the stars immediately behind it by a factor q. The occurrence of clouds with a transparency factor q is, moreover, supposed to be governed by a frequency distribution $\psi(q)$. If so, the authors show that the required probability distribution g can be obtained by solving an integral equation of the form

$$g(u,\xi)+\frac{\partial g}{\partial u}+\frac{\partial g}{\partial \xi}=\int_0^1 g\left(\frac{u}{q},\xi\right)\psi(q)\frac{dq}{q},$$

where u is the observed brightness measured in suitable units, and ξ is the average number of clouds in the direction of the line of sight. It is shown that the foregoing equation enables us to obtain explicit formulae for all the moments of g as functions of ξ and of the moments of $\psi(q)$. An application to the observed structure of the Milky Way is carried out in an attempt to derive the mean and mean-square deviation of the optical thickness of the interstellar clouds.

Z. Kopal (Cambridge, Mass.).

Chandrasekhar, S., and Münch, G. The theory of the fluctuations in brightness of the Milky Way. II. Astrophys. J. **112**, 393–398 (1950).

The integral equation governing the fluctuation in brightness of stellar systems consisting of stars interspersed with obscuring clouds of interstellar matter, deduced previously by the authors [see the preceding review], is applied to the case in which the system extends to infinity in the direction of the line of sight. The respective equation is explicitly solved for the case in which all clouds are equally transparent, and also for one in which the frequency of occurrence of clouds with a transparency factor q is $(n+1)q^n$, where $n(s)$ is a chance variable expressing the number of clouds encountered along the line of sight up to a distance s. The derived distributions of brightness are illustrated.

Z. Kopal (Cambridge, Mass.).

Chandrasekhar, S., and Münch, G. On stellar statistics. Astrophys. J. **113**, 150–165 (1951).

The authors propose a new method for treating the effects of a heterogeneous nature of interstellar absorption on the character of solutions of the fundamental equations of stellar statistics, expressing the mean parallax $\pi(m)$ and the number $\Lambda(m)$ of stars of apparent magnitude m per unit solid angle in various parts of the sky in the form $\Lambda(m)=\int_0^\infty D(s)\phi(M)s^2ds$ and $\Lambda(m)\pi(m)=\int_0^\infty D(s)\phi(M)sds$, where the function $D(s)$ denotes the spatial density of stars at a distance s from the observer, and $\phi(M)$ is the mean luminosity-function of the group of stars under consideration of the argument $M=m+5-5\log s-a(s)$, the function $a(s)$ giving the absorption (in magnitudes) of obscuring matter between the observer and the star. Contrary to previous investigators who usually considered $a(s)$ to be proportional to s, the authors allow $a(s)$ to be a random function which is subject to fluctuation. A numerical inversion of the foregoing integral equations proved to be impossible on account of a relatively low precision of pertinent observational data; but theoretical formulae have been derived for the dispersions to be expected in the observed numbers $N(m)$ of stars, per unit solid angle, brighter than a given apparent magnitude m, counted at different galactic latitudes. The results of star counts tabulated by van Rhijn and by Baker and Kiefer are analyzed in terms of these formulae, and a theoretical prediction based on them is verified. *Z. Kopal* (Cambridge, Mass.).

MECHANICS

Schmid, W. Über die Koppelkurve des Schubkurbelgetriebes. Z. Angew. Math. Mech. **30**, 388–390 (1950).

A study is made of the geometrical properties of the fourth degree curve traced by a point carried by the connecting-rod of a crosshead mechanism. For example, it is shown that the two double points of the curve lie on a line through the center of rotation of the crank. This line has the same angle with the normal to the crosshead track as the angle between the two lines joining the carried point to the pivots of the connecting-rod. *M. Goldberg.*

Noskov, N. I. Spherical construction of positions of a spatial seven-bar linkage. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **9**, no. 33, 53–71 (1950). (Russian)

The mechanism is a seven-link linkage with seven turning pairs. The problem is to determine the positions of all the links if the positions of two of them are given. The method is one of graphical interpolation (projections on two perpendicular planes are used for graphical constructions). Geometric loci are indicated by means of rows of points, and their intersection determined by interpolation. These geometric loci are essentially the sets of positions of an axis A_5 of an open chain $A_1A_2A_3A_4A_5$ when A_1 is fixed, and of the open chain $A_5A_6A_7$ when A_7 is fixed. The method is theoretically trivial and practically worthless because of the admittedly huge volume of drafting involved (50 hours). The analytic method of Dimentberg [same Trudy **5**, no. 17, 5–39 (1948); these Rev. **12**, 549], in the reviewer's opinion, supersedes this paper completely. *A. W. Wundheiler.*

Chang, Ching-Hsian. Verallgemeinerung der einfachen rueckkehrenden Stirnradumlaufgetrieben. Acad. Sinica Science Record **2**, 417–423 (1949).

A graphical and exploratory study is made of the modifications of planetary differential gear drives to produce non-uniform outputs from uniform inputs. The modifications consist of the replacement of gear connections by linkage connections, adding idler gears with linkage connections or the insertion of cams and Geneva mechanisms. The types of output include periodic reversal of motions with rest periods. *M. Goldberg* (Washington, D. C.).

Dimentberg, F. M. On a spatial string polygon. Akad. Nauk SSSR. Inženernyĭ Sbornik **5**, no. 1, 158–162 (1948). (Russian)

The author represents a force system by means of a "cross" consisting of one vector in a "horizontal" plane and one "vertical" vector [Bull. Acad. Sci. URSS. [Izvestiya Akad. Nauk SSSR] Cl. Sci. Tech. **1939**, no. 7, 53–72]. If $R_1, \cdots, R_n$ is a sequence of crosses, $S_0, S_1, \cdots, S_n$ is a "spatial string polygon" of it if $R_i+S_{i-1}=S_i$. The horizontal components of all the crosses inherit the same relationship. Crosses are reciprocal if the relative moment of the corresponding force systems is zero. The homologous sides of two plane string polygons belonging to the same force system intersect on the same straight line. This familiar theorem is generalized as follows: For p (<6) spatial string polygons ${}^kS_i$ ($k=1, \cdots, p$) of the same cross system there exist $p-1$ crosses reciprocal with every cross C_i reciprocal with each of the p homologous crosses ${}^kS_i$ (of same i). This theorem is applied to the analysis of a three-hinge arch.

A. W. Wundheiler (Chicago, Ill.).

Duncan, W. J. The characteristics of systems which are nearly in a state of neutral static stability. Quart J. Mech. Appl. Math. **3**, 452–458 (1950).

A discussion is given of the dynamics of dissipative systems which are nearly in a state of neutral static equilibrium. Formulas are derived for determining, from a knowledge of the neutral mode, a first approximation to the characteristic root and a corresponding modal column. Formulas are also derived for determining successively improved approximations from earlier approximations. A numerical example shows rapid convergence of the successive approximations. The reduction of a nearly neutral system to a neutral system, for application of the method, is discussed. It is pointed out that continuous systems can be analyzed by the method presented if use is made of a finite number of generalized coordinates to describe the motion. *S. Levy.*

Eden, R. J. The Hamiltonian dynamics of non-holonomic systems. Proc. Roy. Soc. London. Ser. A. **205**, 564–583 (1951).

In order to express the classical equations of motion of a nonholonomic system in Poisson bracket form, a set of canonical variables of a system with the same Lagrangian but without constraints is introduced. These auxiliary variables possess instantaneously the same values as the coordinates and momenta of the actual system. Poisson brackets may then be formed with respect to this set of variables. Hamilton's principle and the Hamilton-Jacobi equation are considered and it is shown that the principal function is nonintegrable. *H. C. Corben* (Pittsburgh, Pa.).

Chazy, Jean. Solutions périodiques de la première sorte du problème des trois corps. Bull. Astr. (2) **14**, 153–175 (1949).

This paper is in many respects a repetition of Poincaré's discussion of periodic solutions of the first kind for the three-body problem. The major difference is that a non-canonical set of variables, also due to Poincaré, is used. It is shown that, for a certain domain of the variables chosen, the right hand members of the differential equations can be developed in convergent series in powers of these variables, provided that the masses of the second and third bodies are sufficiently small compared with the mass of the first. Beyond this, the discussion proceeds along the same lines as Poincaré's proof, but in considerably more detail.

D. Brouwer (New Haven, Conn.).

***Delachet, A., et Taillé, J. La balistique.** Presses Universitaires de France, Paris, 1951. 128 pp.

This informative little booklet is for the general reader and for the novice in the science and art of ballistics. As such it touches interestingly on most of the less technical topics within the field, from the earliest history to the era of rocket missiles. The discussion is enriched by numerous well-chosen quantitative data of popular interest and by over forty appropriate diagrams, including a map showing points of impact of V-2 rockets aimed at Paris in 1944. While equations of interior and exterior ballistics are treated briefly, mathematical techniques are not discussed. The reader is referred to other cited sources for original material.

A. A. Bennett (Providence, R. I.).

Hydrodynamics, Aerodynamics, Acoustics

Borg, S. F. On an application of dimensional analysis. Amer. J. Phys. 19, 69–73 (1951).

It is noted that the general form of various equations, in the mechanics of continua, can be partly deduced from an inspection of tensor invariants and their physical dimension. The discussion is brief; thus the formula for capillary pressure jump, $p=T/r+T/r'$, could be equally $p=T/(rr')^{\frac{1}{2}}$, were it not for a tacit linearity hypothesis.

G. Birkhoff (Cambridge, Mass.).

Aržanyh, I. S. A vortex interpretation of the theory of functions of a complex variable. Doklady Akad. Nauk SSSR (N.S.) 73, 667–670 (1950). (Russian)

Let $F(\zeta)=\varphi+i\psi$, $\zeta=x+iy$, be an analytic function interpreted as the complex potential of a steady incompressible flow. The author observes that this flow may be transformed into a rotational three-dimensional flow by assuming a z-component of the velocity vector $\theta(\psi)$, where θ is an arbitrary function. The vorticity $(\Omega_x, \Omega_y, \Omega_z)$ of this flow is given by $\Omega_x-i\Omega_y=F'\theta'$, $\Omega_z=0$. The stream surfaces are given by (1) $\psi=\text{const.}$, (2) $z=\theta(\psi)\int\{|d\zeta/dF|^2dF\}_{\psi=\text{const.}}+\omega(\psi)$, where ω is an arbitrary function. The author calls (1) the equation of the wing and (2) that of the fuselage. For a given $\psi(x,y)=a$ an equation $z=V(x,y)+c$ will represent a possible fuselage if and only if (*) $(\partial/\partial(x,y))\{\partial(V,\psi)/\partial(x,y),\psi\}=0$. The totality of all fuselages obtained in this way is described by a complicated partial differential equation obtained from (*) by eliminating ψ by means of Laplace's equation. Similar considerations are carried out for the unsteady case, $F=F(\zeta,t)$.

L. Bers (Los Angeles, Calif.).

Golubeva, O. V. The equations of two-dimensional motion of an ideal fluid on a curvilinear surface and their application in the theory of filtration. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 287–294 (1950). (Russian)

The author writes (in Gaussian coordinates) the equations of a two-dimensional steady motion of a perfect fluid for the case when the fluid moves along a given surface. Similarly she rewrites the equations of filtration. Under suitable assumption she ends up with Beltrami's equations and obtains some particular solutions (said to be of importance for filtration theory) for the case when the surface is one of revolution, so that the conformal mapping on a plane can be accomplished by quadratures.

L. Bers.

Riegels, F. Berichtigung zu meiner Arbeit in Bd. XVII, S. 94 des Ingenieur-Arch.: "Das Umströmungsproblem bei inkompressiblen Potentialstromungen (II)". Ing.-Arch. 18, 329 (1950).

Cf. Ing.-Arch. 17, 94–106 (1949); these Rev. 11, 274.

Paničkin, I. A. The influence of the boundaries of a free jet of elliptic section on the aerodynamic characteristics of a wing. Akad. Nauk SSSR. Inženernyĭ Sbornik 4, no. 2, 161–173 (1948). (Russian)

The function $T(z)=c\ \mathrm{sn}\ (2\pi^{-1}K \arcsin z/c)$ maps a slit ellipse in the $z=x+iy$ plane with foci $z=\pm c$ into a slit circle of radius $R=ck^{-\frac{1}{2}}$ in the $T=\xi+i\eta$ plane. The real period $4K$ and the modulus k of the Jacobian elliptic function are known functions of the ratio b/a of the lengths of the minor and major axes. The function $T(z)$ transforms the perturbation potential $\varphi(x,y)$ of a given wing on $|x|\leqq s$, $y=0$ in an elliptical open throat wind tunnel into the perturbation potential $\varphi_k(\xi,\eta)$ of some wing on $|\xi|\leqq s_k=T(s)$, $\eta=0$ in a circular jet. At corresponding points of the two wings the circulations and downwash velocities satisfy (1) $\Gamma(x)=\Gamma_k(\xi)$, and (2) $w_0(x)+w_1(x)=(w_{0k}(\xi)+w_{1k}(\xi))d\xi/dx$, where w_0 and w_{0k}, corresponding to wings in unbounded streams, are calculated by standard lifting line methods for (3) $\Gamma(x)=\Gamma_0(1-(x/s)^2)^{\frac{1}{2}}\sum A_n'(x/s)^{2n}$, $|x|\leqq s$, and (4) $\Gamma_k(\xi)=\Gamma_0(1-(\xi/s_k)^2)^{\frac{1}{2}}\sum A_n(\xi/s_k)^{2n}$, $|\xi|\leqq s_k$. The term w_{1k}, due to the influence of the circular boundary, is obtained by assigning vortex density $-d\Gamma_k(\xi)/d\xi$ to $T=R^2/\xi$, so chosen to satisfy the boundary condition $\varphi_k=\text{constant}$ on $|T|=R$. Then $\int_{-s}^{+s}(w_0+w_1)\Gamma dx=\int_{-s_k}^{+s_k}(w_{0k}+w_{1k})\Gamma_k d\xi$ and (1) yield corrections for induced drag and for downwash angle (based on a spanwise average of w_1) in the elliptical jet as functions of A_n and A_n'. Expanding both members of (1) into power series in x/s yields the relations between A_n and A_n'. Formulas for A_1, A_2, A_3 when $A_n'=0$, $n\geqq 2$, have been applied to compute the variation of these corrections with varying b/a and s/a. The author's treatment of the problem differs from the cited work of Sanuki and Tani [Proc. Phys.-Math. Soc. Japan (3) 14, 592–603 (1932)] and Rosenhead [Proc. Roy. Soc. London. Ser. A, 140, 579–604 (1933)] in the consideration of more general forms (3) and (4) for the circulation.

J. H. Giese (Havre de Grace, Md.).

Paničkin, I. A. The determination of the spanwise distribution of circulation of a wing in a closed stream of rectangular cross-section. Akad. Nauk SSSR. Inženernyĭ Sbornik 5, no. 1, 189–197 (1948). (Russian)

The function $\zeta(z)=a\ \mathrm{sn}\ (Kz/a)$ maps the rectangle $|x|\leqq a$, $|y|\leqq h$ in the $z=x+iy$ plane onto a slit circle $|\zeta|\leqq R=ak^{-\frac{1}{2}}$ in the $\zeta=\xi+i\eta$ plane. The modulus k and real period $4K$ of the Jacobian elliptic function are known functions of $\epsilon=h/a$. Moreover, ζ maps the downwash field of a given rectangular wing on $x=0$, $|y|\leqq s<h$ in a closed rectangular wind tunnel into the downwash field of a wing on $\xi=0$, $|\eta|\leqq s_1=-ia\zeta(is)$ in a closed circular tunnel. At corresponding points of these wings the circulations and downwash velocities satisfy $\Gamma(y)=\Gamma_1(\eta)$ and $w(y)=w_1(\eta)d\eta/dy$. In the circular tunnel $4\pi w_1(\eta)=\int_{-s_1}^{+s_1}(d\Gamma_1(u)/du)[(u-\eta)^{-1}-u(R^2-u\eta)^{-1}]du$. Hence $\Gamma(y)=\frac{1}{2}a_0b[\alpha_0V+w(y)]$ yields an integro-differential equation for $\Gamma_1(\eta)$, where a_0 is the lift coefficient slope, b wing chord, α_0 geometrical angle of attack, and V the speed at infinity. A search for solutions of the form $\Gamma_1(\eta)=\frac{1}{2}a_0b\alpha_0V\sum_0^\infty G_n \sin(2n+1)\varphi$, where $\cos\varphi=-\eta/s_1$, leads to an infinite system of linear equations for G_n. For a wing of aspect ratio $2s/b=8$ in two tunnels with $s/a=0.8$ and $\epsilon=1$ or 1.5, $\Gamma(y)$ has been calculated by neglecting G_n, $n\geqq 6$, and compared with $\Gamma(y)$ for an unbounded stream.

J. H. Giese (Havre de Grace, Md.).

Paničkin, I. A. On the downwash behind a wing. Akad. Nauk SSSR. Inženernyĭ Sbornik 5, no. 2, 164–170 (1949). (Russian)

The downwash angle has been calculated by lifting line theory for linearized subsonic flow. Computations for a single horseshoe vortex show very slight dependence on Mach number for $M\leqq 0.6$. For the general spanwise circulation distribution $\Gamma(\zeta)=\Gamma_0(1-(\zeta/s)^2)^{\frac{1}{2}}\sum A_n(\zeta/s)^{2n}$ on the segment of the ζ-axis $|\zeta|\leqq s$ (the semi-span), where Γ_0 and A_n are constants, the downwash angle at $(\xi,0,0)$ is $-(\Gamma_0/2\pi V)\sum a_n[(k/\xi)F_{n+1}+(2Ns)^{-1}(\pi p_n+2\xi kF_n/Ns)]$, where V is the speed at infinity, $a_n=(2n+1)A_n-(2n+2)A_{n+1}$, $k=(1+\xi^2/N^2s^2)^{-\frac{1}{2}}$, $N^2=1-M^2$, and $p_n=(2n)!/2^{2n}(n!)^2$. The F_n satisfy $(2n-1)k'^2F_{n-1}+2n(k^2-k'^2)F_n-(2n+1)k^2F_{n+1}=0$, where $k'^2=1-k^2$, and F_0 and $k^2F_1+k'^2F_0$ are the complete elliptic integrals of the first and second kind, respectively.

of modulus k. The F_n ($n \leqq 3$) have been tabulated for $\xi/Ns = 0.25m$, $1 \leqq m \leqq 8$. *J. H. Giese.*

Tison, L.-J. Le déversoir à seuil épais. Houille Blanche **5**, 426–439 (1950).

The author makes a number of theoretical and experimental contributions to the study of the hydraulics of the broadcrested weir. Among others, for the case of the standing wave, the author gives both experimental data and a new theory, for the wavelength computation. This theory, which is an alternative to Boussinesq's well-known theory of this phenomenon, is based on rather arbitrary assumptions; its results, within the rather narrow range of discharges investigated, nevertheless agree surprisingly well with the author's experimental findings. *P. Neményi.*

Galin, L. A. On unsteady filtration with constant pressure on the boundary. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 111–116 (1951). (Russian)

This paper is an extension of the problem treated earlier by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 246–249 (1945); these Rev. **7**, 229]. The extension consists in admitting more general regions and more general functions in terms of which the initial conditions of the problem are given. The method used for finding particular solutions consists in assuming a solution of a special form, substituting it into the nonlinear differential equation of the problem, and imposing further sufficient conditions on the separate terms of the equation to satisfy it. *H. P. Thielman.*

Kufarev, P. P. Solution of a problem on the contour of the oil-bearing region for lodes with a chain of gaps. Doklady Akad. Nauk SSSR (N.S.) **75**, 353–355 (1950). (Russian)

Soit $|\Re z| < \frac{1}{2}\omega_1^0$ une bande infinie représentant l'état initial d'un gisement petrolifère. Cette bande porte un suite infinie des puits d'exploitation équidistante (affixe $= 2n\omega_2^0 = 2nih$) à débit $2\pi q(t)$ égal pour tous les puits. L'auteur étudie la variation des limites de ce gisement en se servant des fonctions elliptiques de Weierstrass $p(u; 2\omega_1, 2\omega_2)$, $\zeta(u)$, $\sigma(u)$ pour obtenir la représentation conforme de l'état initial du gisement sur son état au moment t. En désignant par u la variable complexe dans cette région on a la transformation cherchée sous la forme $z = \beta(t)u + A(t)\zeta(u-\omega_1) + A(t)\eta_1$, les fonctions $\beta(t)$, $A(t)$, $\eta_1 = \zeta(\omega_1)$ étant des solutions de certaines équations différentielles ordinaires.

V. A. Kostitzin (Paris).

Kufarev, P. P. The problem of the contour of the oil-bearing region for a circle with an arbitrary number of gaps. Doklady Akad. Nauk SSSR (N.S.) **75**, 507–510 (1950). (Russian)

Dans un précédent article [mêmes Doklady (N.S.) **60**, 1333–1334 (1948); ces Rev. **10**, 241] l'auteur a donné une solution du problème de la réduction du contour d'un gisement petrolifère dans le cas de forme initiale circulaire et d'un seul puits d'exploitation. L'auteur donne actuellement une généralisation de ses résultats en supposant le nombre des puits et leur distribution sur la surface arbitraires. Dans ce but il utilise la représentation conforme de l'état initial sur l'état du gisement au moment t. La solution a la forme $z = \beta(t)w + \alpha(t) + \sum_{k=1}^{n} A_k(t)/(w - \alpha_k(t))$, les fonctions α_k, β, A_k étant solutions d'un système d'équations algébrique. Dans le cas particulier de disposition symétrique des puits d'égale puissance la solution se calcule facilement.

V. A. Kostitzin (Paris).

Dean, W. R. Slow motion of viscous liquid in a semi-infinite channel. Proc. Cambridge Philos. Soc. **47**, 127–141 (1951).

A slow two-dimensional steady motion of a viscous fluid under pressure in a semi-infinite channel is considered. A form of solution for the stream function is first chosen so that it readily reduces to constant values on the walls. To satisfy also the conditions that its normal derivative vanishes on the boundaries, two infinite series are introduced. It is found, however, that the no-slip conditions can only be approximately satisfied. At great distance from the entrance, outside the channel, the flow is approximately radial; whereas, inside the channel far downstream, the flow becomes that of an infinite channel with constant pressure gradient. The pressure along the center line was plotted. It shows that the pressure increases from a reference value at negative infinity and approaches asymptotically to that of uniform gradient at positive infinity.

Y. H. Kuo (Ithaca, N. Y.).

Žukov, A. M. On the motion in a plane of a sliding block of rectangular form. Akad. Nauk SSSR. Inženernyĭ Sbornik **5**, no. 1, 179–181 (1948). (Russian)

Le problème du mouvement d'un bloque rectangulaire plan a été étudié par Michell. L'auteur étudie le cas ou la surface inférieure est légèrement déformée et cherche à déterminer l'influence de cette petite déformation sur la pression et sur la distribution de la pression du lubrifiant sur la surface. L'auteur utilise la loi de Reynolds pour définir la pression du bloque pour des conditions aux limites données. En supposant la surface intérieure cylindrique, le coefficient de viscosité du lubrifiant constant et son épaisseur petite, l'auteur arrive pour la pression à une équation aux dérivées partielles du type elliptique dont il donne la solution sous forme d'une série. En supposant un bloque de dimensions $\pi \times \pi$ le second terme est de l'ordre de 10^{-4}. On calcule aisément la pression totale, la résistance totale, et le centre d'application de la force. *M. Kiveliovitch* (Paris).

Bogert, B. P. Classical viscosity in tubes and cavities of large dimensions. J. Acoust. Soc. Amer. **22**, 432–437 (1950).

The author treats some approximate solutions for wave propagation in cylindrical and rectangular tubes. He assumes a Rayleigh dissipation function F. Let $\mathfrak{A}$ denote time average over a cycle so $P_{\mathfrak{A}}$ is the averaged power transmitted. Accordingly, neglecting all other losses (1) $-\partial P_{\mathfrak{A}}/\partial z = \partial F_{\mathfrak{A}}/\partial z$, where z is longitudinal distance. The author's method consists of using (1) together with the assumption that the longitudinal velocity is $e^{-\beta z}\psi$, where ψ has no longitudinal damping terms. Then (1) leads to a value for β in terms of other parameters. For instance, in a cylindrical tube with zero radial velocity, ψ is the sum of a sinusoidal wave and a damped sinusoidal wave with damping term $e^{-\alpha s}$, where s is the radial distance from the periphery and $\alpha = (\omega\rho_0/2\mu)^{\frac{1}{2}}$ in a standard notation. On neglecting higher α and β powers the result for β is that arrived at by other methods as well. Similar calculations are given for a rectangular tube and for a circular tube with radial velocity.

D. G. Bourgin (Urbana, Ill.).

Thiriot, K.-H. Grenzschichtströmung kurz nach dem plötzlichen Anlauf bzw. Abstoppen eines rotierenden Bodens. Z. Angew. Math. Mech. **30**, 390–393 (1950).

In this note, nonstationary laminar boundary-layer flow over a rotating disc is considered for two special cases,

namely, (1) a disc in a viscous fluid suddenly starts to rotate with a constant angular velocity ω and (2) a uniformly rotating disc is suddenly brought to stop. Let (v, u, w) be the velocity components in cylindrical coordinates (r, φ, z) and z the axis of rotation. The boundary conditions for the second problem are: for the time $t\geqq 0$ when $z=0, v=u=w=0$ and $z=\infty$, $u=r\omega$, $v=0$, and these apply also for the first problem when u is redefined to suppress the rotation of the disc; the form of solutions in both cases is:

$$u=\omega r[F_0(\eta)+(\omega t)^2F_2(\eta)+\cdots],$$
$$v=\omega r[(\omega t)G_1(\eta)+(\omega t)^3G_3(\eta)+\cdots]$$

with $\eta=z/2(\nu t)^{\frac{1}{2}}$. The functions F_0, F_1, G_1, and G_3 have been integrated and tabulated for both problems. The author mentions that his earlier paper on this same subject [same Z. **20**, 1–13 (1940)] was in error, as pointed out by Görtler [Naturforschung und Medizin in Deutschland 1939–1946, Band 5, pp. 33–73, Dieterich, Weisbaden, 1948; these Rev. **11**, 221]. *Y. H. Kuo* (Ithaca, N. Y.).

Lessen, Martin. On stability of free laminar boundary layer between parallel streams. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **979**, 9 pp. (1950).

Cf. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1929** (1949); these Rev. **11**, 697.

Burgers, J. M. Correlation problems in a one-dimensional model of turbulence. II. Nederl. Akad. Wetensch., Proc. **53**, 393–406 (1950).

[For part I see same vol., 247–260 (1950); these Rev. **11**, 752.] In this part, the author studies the double and triple correlations in his one-dimensional model of turbulence. Again, several interesting properties in the usual theories are reproduced, e.g., the Loitsiansky invariant. The dynamical equations are used to calculate the equation of propagation of the double correlation function.
C. C. Lin (Cambridge, Mass.).

Burgers, J. M. Correlation problems in a one-dimensional model of turbulence. III. Nederl. Akad. Wetensch., Proc. **53**, 718–731 (1950).
Burgers, J. M. Correlation problems in a one-dimensional model of turbulence. IV. Nederl. Akad. Wetensch., Proc. **53**, 732–742 (1950).

In these two parts, the author continues his study [see the preceding review and reference cited there] of a one-dimensional model of turbulence. In part III, investigations are made of the double and triple correlations, equation for the propagation of correlation, and other approaches used in the conventional treatment of the problem of turbulence. It is shown that such concepts do not allow us to predict the shape of the correlation function even with the assumption of its self-preserving character. It is therefore concluded that another statistical treatment, of a more basic nature, is needed. The author then proceeds to indicate such a treatment with some preliminary results (part IV).
C. C. Lin (Cambridge, Mass.).

Burgers, J. M. On a correlation problem in a model of turbulence. Ricerca Sci. **20**, 1933–1937 (1950).

Lecture given at the Colloque Internationale de Mécanique, Pallanza, June, 1950; cf. the three preceding papers and reference cited there.

Güttinger, Werner. Der Stosseffekt auf eine Flüssigkeitskugel als Grundlage einer physikalischen Theorie der Entstehung von Gehirnverletzungen. Z. Naturforschung **5a**, 622–628 (1950).

This paper is an attempt at a theoretical analysis of the physical factors contributing to brain injury when the skull is accelerated rapidly, as when struck a blow. The theory focuses on conditions for occurrence of dilatations in the brain material (assumed to be a homogeneous, slightly compressible fluid), since dilatations are presumed the immediate cause of brain lesions in this type of injury. The analysis is idealized by assuming the skull is spherical and essentially rigid, that the wave equation governs the propagation of pressures in the brain fluid filling the sphere, and that the acceleration of the skull is rectilinear. From knowledge of the form of the impulse, the relevant mixed initial and boundary value problem for the wave equation is solved by methods which have become standard in acoustics. The author examines the pressure waves in the brain fluid following impulses of simple shape and different durations and arrives at qualitative conditions that brain injury occur at either pole of the sphere or in the interior. Among the results is the interesting fact that if the head is struck a (not too severe) blow, damage to the brain occurs in general at the pole opposite and not at the point of contact.
D. Gilbarg (Bloomington, Ind.).

Perl, W., and Klein, Milton M. Theoretical investigation of transonic similarity for bodies of revolution. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2239**, 32 pp. (1950).

An iteration method is applied to the solution of the partial differential equation for axially symmetric compressible flow past slender bodies of revolution. In this approximation procedure, the only terms retained are those of the highest-order magnitude as the lateral distance ratio and the compressibility factor $(1-M^2)^{\frac{1}{2}}$ approach zero. It is concluded from the character of the approximations that any transonic similarity law close to the body is probably valid only for extremely slender bodies not of practical interest. A different similarity law seems to hold in the field of flow not close to the body. These results are rather different from those of von Kármán [J. Math. Physics **26**, 182–190 (1947); these Rev. **9**, 217]. An attempt is made to reconcile these differences for the field of flow at some distance from the body, but there appears to be no agreement between the results obtained for the flow at the body itself.
E. N. Nilson (Hartford, Conn.).

Walters, A. G. On the propagation of disturbances from moving sources. Proc. Cambridge Philos. Soc. **47**, 109–126 (1951).

In an earlier paper [same Proc. **45**, 69–80 (1949); these Rev. **10**, 196] the author related the "Green's vibrational function" of certain time-dependent linear partial differential equations through a Laplace transformation to the Green's function of the associated steady state equations. In this paper he extends the idea of the Green's vibrational function to equations of the form $D(V)=b^{-2}(\partial^2V/\partial t^2+a\,\partial V/\partial t)$. The concept of source of a disturbance is defined with the help of the Green's vibrational function using standard integral superposition methods, and is applied to several physical problems, among them the Doppler effect for subsonic and supersonic sources, and the first and higher approximation for the plane supersonic flow of a compressible fluid past a slender profile. *D. Gilbarg* (Bloomington, Ind.).

Nabarro, F. R. N. The force acting on a body moving uniformly through a gas containing sound waves. Philos. Mag. (7) 41, 1270–1280 (1950).

In dealing with the problem of interaction of sound waves with a body moving with a uniform velocity, the author shows that if there is a single sound wave, the force acting on the body is zero. However, if the body moves in an isotropic distribution of sound waves, which are anisotropic in coordinates moving with the body, the force will be proportional to the Mach number and the energy density of sound waves. The effect is mostly due to distribution of sound with respect to the moving body and the apparent change of wave-length. It is shown that inclusion of nonlinearity in the equation produces no force. *Y. H. Kuo.*

Martin, J. C. Retarded potentials of supersonic flow. Quart. Appl. Math. 8, 358–364 (1951).

Whitehead, L. G., Wu, L. Y., and Waters, M. H. L. Contracting ducts of finite length. Aeronaut. Quart. 2, 254–271 (1951).

Taylor, J. Lockwood. An analysis of the lift on straight, yawed and swept-back wings. Aeronaut. Quart. 2, 293–304 (1951).

Heaslet, Max A., and Lomax, Harvard. The application of Green's theorem to the solution of boundary-value problems in linearized supersonic wing theory. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **961**, 14 pp. (1950).

Cf. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1767** (1949); these Rev. **10**, 753.

Harmon, Sidney M. Correspondence flows for wings in linearized potential fields at subsonic and supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2303, 29 pp. (1951).

Miles, John W. The oscillating rectangular airfoil at supersonic speeds. Quart. Appl. Math. 9, 47–65 (1951).

This paper appeared earlier as a report [Naval Ordnance Test Station, Inyokern, Calif. Tech. Memo. **RRB-15** (1949); these Rev. **12**, 217].

Friedman, Raymond, and Burke, Edward. On the one-dimensional theory of flame structure. J. Aeronaut. Sci. 18, 239–246 (1951).

The most difficult problem in the theory of flame propagation is the chemical kinetics of the reaction. The purpose of this paper is, however, not the solution of this problem, but rather an investigation of the effects on the temperature and concentration distribution inside the flame zone, caused by a change in the spatial distribution of the rate of energy liberation and the rate of species formation, and by the variation of the heat conduction coefficient and the diffusion coefficient with temperature. For three cases studied, the energy liberation and the species formation are assumed to be limited in a small zone called the thickness of the flame. The distributions are respectively, rectangular, triangular with maximum at the end of the zone, and a single sine wave with maximum at the middle of the zone. Preceding the reaction zone is the preheat zone. The fourth case investigated has an exponential distribution of the rates. With the usual simplification of the energy equation and concentration equation, the solutions are obtained by simple quadrature. The results show that the influences of the variations of the shape of rate distributions and the gas constants are not large. The authors suggest that the nondimensional relations obtained can be used to check the consistency of any one set of experimental data. In the appendix, the authors give an interesting molecular interpretation of the results and arrive at the conclusion that the inverse of the square of Mach number of the flame speed is of the order of the number of molecular collisions necessary to complete the reaction in the flame. *H. S. Tsien.*

Kuo, Hsiao-lan. The motion of atmospheric vortices and the general circulation. J. Meteorol. 7, 247–258 (1950).

C'est un fait bien connu que le mouvement de la troposphère supérieure est caractérisé par la présence de gros centres cycloniques au Nord et de centres anticycloniques au Sud, séparés par une bande de forts courants d'Ouest. Afin d'expliquer cette distribution des tourbillons, l'auteur étudie en partant des équations classiques du mouvement, deux types de tourbillons. Pour le premier il suppose le fluide en mouvement horizontal, non divergent et barotropique. Dans ce cas la composante verticale du mouvement tourbillonnaire absolu est invariable et le tourbillon peut être assimilé à un tourbillon à parois solides contenant toujours la même masse de fluide. L'auteur étudie dans ce cas particulier le mouvement d'un tourbillon circulaire. Dans le cas général du mouvement de l'atmosphère, la constance de la composante verticale est rarement observée; elle a plutôt tendance à croître en allant vers le Nord. L'auteur étudie donc un autre modèle de tourbillon en introduisant les effets de la divergence, du frottement à la surface, et de la sphéricité de la terre. L'auteur croit que ces modèles de tourbillons atmosphériques et les mouvements qui en résultent peuvent être utiles dans l'explication de l'origine et de la nature de la circulation générale.

M. Kiveliovitch (Paris).

Sherman, Leon. On permanent perturbations of zonal atmospheric motion, with applications to low latitudes. J. Meteorol. 8, 84–94 (1951).

The components of the vorticity equation and the equation of continuity are treated as a system of 4 equations in the 4 unknowns, the density and the three velocity components. Friction and quadratic terms in the velocities are neglected. Two kinds of solutions are found, one in which the vertical motion is known and the other in which the vertical motion can be computed from measurable variables. The first type of solution is used for the construction of models of atmospheric motion, the second accounts for the observed relation between meridianal and vertical motion.

H. Panofsky (New York, N. Y.).

Van Mieghem, Jacques. Les bilans de la quantité de mouvement et de l'énergie mécanique dans l'atmosphère. Ann. Géophysique 6, 227–237 (1950).

The author rediscusses the equations expressing the angular momentum and kinetic energy budgets of the atmosphere. He concludes that one should not consider the kinetic energy budget from the equation of motion alone, because then the terms expressing conversion of the energy depend on the system of reference; moreover kinetic energy is treated as a vector quantity. *H. Panofsky.*

Pekeris, C. L. Free oscillations of an atmosphere in which temperature increases linearly with height. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2209**, 24 pp. (1950).

Pour une atmosphère stratifiée et une distribution des températures en altitude donnée d'avance, Taylor [Proc.

Roy. Soc. London. Ser. A. **156**, 318–326 (1936)] a déduit une équation différentielle de second ordre à laquelle doit satisfaire la divergence χ du mouvement ondulatoire. L'auteur montre que si, pour les grandes altitudes, la température croît linéairement avec l'altitude, les deux solutions nécessitent une croissance infinie de l'énergie. Afin d'éviter cette incohérence l'auteur a établi une équation plus générale de χ [ibid. **158**, 650–671 (1937)] qu'il résout à l'aide de la fonction W de Whittaker. Pour les grandes longueurs d'onde la vitesse de propagation ne tend pas vers une limite, comme dans le cas d'un gradient de température négatif, ou zéro, mais croît linéairement avec la période. Il en est de même de la vitesse de groupe qui tend vers la moitié de la vitesse de phase. Dans le cas d'un gradient de température négatif ou zéro, la marée atmosphérique est analogue à celle d'un océan uniforme d'une hauteur équivalente H (d'environ 8 km et qui dépend de la distribution donnée des températures). Dans le cas d'un gradient de température positif, la marée atmosphérique est radicalement différente et dans le cas analogue de la marée océanique la profondeur est fonction de la période et croît indéfiniment avec la période. Il en est de même de la propagation acoustique qui diffère de la propagation atmosphérique ordinaire. *M. Kiveliovitch.*

Giāo, Antônio. Analysis of the pressure variations at sea-level. Geofis. Pura Appl. **16**, no. 3–4, 20 pp. (1950).

L'auteur se propose de former l'équation de la tendance $\partial p/\partial t$ au niveau de la mer en se basant sur la validité de l'équation statique; la constante du gradient de température dans la troposphère est nulle dans la stratosphère. En utilisant les équations du mouvement, l'auteur arrive en négligeant un certain nombre de termes, pratiquement négligeables, à une équation intégro-différentielle linéaire non homogène. L'auteur obtient quelques solutions particulières de cette équation en attribuant au second membre certaines valeurs choisies d'avance. *M. Kiveliovitch.*

Zaĭcev, L. P., and Zvolinskiĭ, N. V. Investigation of the head wave arising at the boundary between two elastic liquids. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **15**, 20–39 (1951). (Russian)

The authors call a "head" wave that which is known in seismology as a "refracted" wave. They consider the propagation of a disturbance in two dimensions in an infinite medium composed of two liquids, apply the method developed by Smirnov and Sobolev [cf. Sobolev, chapter 12 of the Russian edition of Frank and von Mises, Die Differential- und Integralgleichungen der Mechanik und Physik, ONTI, 1937]. The solutions of the wave equation are represented by the real parts of an arbitrary function of a complex variable θ. This variable is given by the equation $t-\theta x+y(a^{-2}-\theta^2)^{\frac{1}{2}}+p(\theta)=0$, where p is an arbitrary function. On assuming a certain form of velocity potential created by a point (line) source, the boundary conditions are satisfied by a choice of other potentials. The corresponding displacements for the incident, reflected, and "head" wave are derived in an explicit form. Some conditions seem to be given in an incorrect form. *W. S. Jardetzky.*

Četaev, D. N. On the radiation of sound from a piston. Doklady Akad. Nauk SSSR (N.S.) **76**, 813–816 (1951). (Russian)

The author estimates the expression

$$\varphi=(v/2\pi)\int_S\int r^{-1}e^{-ikr}dS,$$

where S is a finite region of the plane $z=0$, this giving the amplitude of the velocity potential of sound waves from an oscillating piston. He expresses φ as the sum of an explicit term and a line-integral over the boundary of S. For large k the integral is approximated to by means of asymptotic series associated with points of stationary phase. Finally, for a rectangular piston an expression is found for the radiation resistance, a numerical table being given for the case in which the piston is square. *F. V. Atkinson* (Ibadan).

Tartakovskiĭ, B. D. Sound transition layers. Doklady Akad. Nauk SSSR (N.S.) **75**, 29–32 (1950). (Russian)

Avec le développement de la technique ultrasonique en acoustique le problème de l'établissement de transparence complète au passage d'un milieu à l'autre devient de plus en plus important. La solution exacte du problème de propagation des ondes planes à travers des couches homogènes obtenue auparavant par l'auteur [mêmes Doklady (N.S.) **71**, 465–468 (1950); ces Rev. **11**, 700] lui permet d'examiner les conditions du passage complet de l'énergie des ondes d'un milieu à l'autre, si on intercale n couches planes homogènes convenablement choisies. L'auteur affirme qu'en principe on peut obtenir une transparence complète quels que soient les milieux, en plaçant entre eux une combinaison des couches planes, et ceci quels que soient les matériaux utilisés. *V. A. Kostitzin* (Paris).

Horton, C. W., and Karal, F. C., Jr. On the diffraction of a plane sound wave by a paraboloid of revolution. J. Acoust. Soc. Amer. **22**, 855–856 (1950).

This paper attacks the problem of diffraction of a plane sound wave by the convex surface of a paraboloid of revolution (a) when the paraboloidal surface separates fluids having different properties, and (b) when the material filling the concave region is perfectly rigid. Pressures are expressed in infinite series of paraboloidal wave functions. Problem (b) is solved completely, but the solution of problem (a) is carried only to an infinite system of linear equations in an infinite number of unknowns. In problem (b) it is shown that when the incident wave is oriented perpendicular to the axis, the ratio of the vertex pressure to the incident plane wave pressure takes an especially simple form. This is plotted in fig. 2. Further numerical results must await the computation of the paraboloidal wave functions. It must be noted that the primes in formulas (10), (13), (15), (16) do not apply to m but denote derivatives of the paraboloidal wave functions. *E. Pinney* (Berkeley, Calif.).

Twersky, Victor. On the scattered reflection of scalar waves from absorbent surfaces. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-22**, ii+iii+47 pp. (1950).

The author's summary is as follows: "The nonspecular reflection of plane waves of sound by various absorbent surfaces composed of either semicylindrical or hemispherical bosses (protuberances) on an infinite plane is analyzed. Approximate solutions in terms of eigenfunctions for the problem of the single boss (with normal impedance Z) on an infinite plane (with normal impedance Z') and a plane wave at an arbitrary angle of incidence are derived through consideration of a cylinder or sphere and two simultaneously incident "image waves." Small finite patterned distributions of such bosses are then treated and the far field solution obtained subject to the restriction that the secondary excitations of the various bosses be neglected. These solutions are found to contain the characteristic Fraunhofer

terms for a grating or lattice. The asymptotic solutions for the single bosses ($Kr\gg 1$, $Ka<1$) are then extended to consider both small finite and infinite uniform random distributions. The solutions for the finite distributions are found to contain the characteristic Fraunhofer terms for similarly shaped apertures. The solutions for the infinite distributions (of semicylinders or hemispheres) are found to be remarkably similar when expressed in terms of the volumetric departure from the plane per cm^2 of distribution. The results obtained for the various limiting cases are then compared in the plane of incidence. For certain ranges of the parameters, the results predict the occurrence of a minimum at the specular angle of reflection and the occurrence of some critical angle of incidence for which the reflection at the specular angle is completely specular. The equivalent problems for cylinders and spheres are also considered."

M. J. O. Strutt (Zurich).

Twersky, Victor. On the non-specular reflection of plane waves of sound. J. Acoust. Soc. Amer. **22**, 539–546 (1950).

Abbreviation of the author's dissertation reviewed above.

M. J. O. Strutt (Zurich).

Mintzer, David. Transient sounds in rooms. J. Acoust. Soc. Amer. **22**, 341–352 (1950).

The author applies Laplace transforms to the problem of the title. He first considers the one-dimensional case in which a plane wave is propagated down a rigid-walled tube from a source at one end toward an acoustic termination at the other end. The velocity potential for an arbitrary particle displaced at the input is found as a series of which each term represents the effect of a reflection from the ends of the tube. In the three-dimensional case a spherical wave from an arbitrary input source is considered, first in an unbounded region and then in regions containing one wall, three perpendicular walls, and two parallel walls. Finally, he considers a point source in a rectangular room. He uses an image method. The results are in the form of a series each of whose terms represents reflections from individual walls as well as reflections between walls. Some sample calculations are made for the one- and three-dimensional systems.

M. J. O. Strutt (Zurich).

Elasticity, Plasticity

***Novožilov, V. V. Osnovy nelineĭnoĭ teorii uprugosti. [Foundations of the Nonlinear Theory of Elasticity].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948. 211 pp.

This book makes a twofold contribution to the available literature on elasticity; in addition to presenting a penetrating account of the modern thinking on the nonlinear theory, it is also a clear account of the fundamentals of the modern mathematical theory as a whole. In fact, one gets the impression that the author's rejection of the usual linearizing assumptions in the derivation of the fundamental relations enhances (rather than encumbers) the reader's comprehension of the geometric and physical assumptions of the theory. In its use of mathematical tools the book falls between the engineering texts and, say, the work of Sokolnikoff and Mushelišvili. The use of tensors is avoided; however, the usually cumbersome component notation is employed with great skill. Except for the language difficulty, the book would make an excellent companion piece to Prescott [Applied Elasticity, Longmans, Green, London, 1924] for use by first and second year graduate students. The present book supplies the physical insight into the theory which is lacking in Prescott, but without the wealth of examples found in the latter work.

The analyses of strain, stress, and of the equilibrium equations are presented in full generality for homogeneous isotropic bodies and the classical relations are obtained by a two-stage modification of the resulting expression. It is shown that the classical formulation contains two assumptions to the effect that (a) the strains and the angles of rotation of the body are small compared to unity, and (b) the products of the angles of rotation are small compared to certain corresponding components of strain. The specific strain energy of a body is represented as a series expansion in terms of the strain invariants and the stress-strain relations are given in terms of the coefficients of this expansion. The specific linearity assumptions underlying Hooke's law are thus made explicit. On the basis of two experimental observations of the behavior of the stress invariants, the same formulation (originally derived for conservative forces) is shown to yield the Hencky stress-strain equations for loading in plastic bodies. Two types of nonlinearity are shown to enter the problems of deformation of elastic bodies. The geometrical nonlinearity results when the angles of rotation of the body and the strains are no longer negligible compared to unity. The physical nonlinearity results when the strains are no longer negligible compared to certain physical constants of the material (proportional limits).

The chapter headings are as follows: (I) Geometry of strain; (II) Equilibrium of a volume element; (III) Strain energy, boundary conditions, stress-strain law; (IV) Formulation of the elastic (boundary value) problem in terms of stresses; (V) The problem of elastic stability; (VI) Deformation of elastic bodies. Chapter (VI) presents applications of the nonlinear theory to the following cases: (a) bending of thin plates and shells; (b) bending and torsion of rods. An extensive bibliography is appended.

H. I. Ansoff.

Aržanyh, I. S. On the theory of integration of the dynamical equations of an isotropic elastic body. Doklady Akad. Nauk SSSR (N.S.) **73**, 41–44 (1950). (Russian)

The present paper is devoted to various questions concerning the integration of the dynamical equations of an isotropic elastic body: (1) $\partial^2 u/\partial t^2=\alpha \operatorname{grad}\operatorname{div} u-\beta \operatorname{rot}\operatorname{rot} u$, where the vector-valued function u is the displacement, and $\alpha=(\lambda+2\mu)/\rho$, $\beta=\mu/\rho$, where λ and μ are Lamé's constants of elasticity, and ρ is the density of the body. The author first gives certain general representations of solutions of (1) in terms of arbitrary functions, which extend known results in the static case (when $\partial^2 u/\partial t^2$ is replaced by zero). Let S denote a smooth surface, and Q denote either its interior or its exterior. If Q is the exterior of S then u is required to be "regular at infinity," in addition to satisfying (1). Using a lemma derived previously, the author derives integral equations which are equivalent to the solution of certain free and forced vibration problems for the body Q.

J. B. Diaz (College Park, Md.).

Bogdanoff, John Lee. On the theory of dislocations. J. Appl. Phys. **21**, 1258–1263 (1950).

The author states that he shows by examples that "dislocations of a more general type than encountered in classical theory are possible." By "classical" theory he means Volterra dislocations, which are characterized by Weingarten's

theorem. He is apparently unaware of the general theory of dislocations given by Somigliana [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **23**$_1$, 463–472 (1914)], among whose results is a characterization of the special class of dislocations described by Weingarten's theorem; a part of Somigliana's results have been given even earlier by Hadamard [Leçons sur la propagation des ondes . . ., Hermann, Paris, 1903, see especially §77 and chapter VI]. The author gives only two references to the extensive literature of the subject of dislocations since 1907 [a survey up to 1930 is given by P. Neményi, Z. Angew. Math. Mech. **11**, 59–70 (1931)], but it is not impossible that his particular solutions may be new. [Reviewer's note: The author's statement in §1 that from the continuity of the traction follows the continuity of those stresses which "enter the expression for the traction on Ω" is obviously false except in the special case when Ω is a coordinate surface.]

C. Truesdell (Bloomington, Ind.).

Radenković, D. Une solution du problème à deux dimensions de la théorie de l'élasticité. Acad. Serbe Sci. Publ. Inst. Math. **3**, 127–136 (1950).

After summarising the general method given by Love [J. London Math. Soc. **3**, 144–156 (1928)] for the determination of a biharmonic function holomorphic in a given plane region and satisfying specified conditions on the boundary of the region, the author goes on to give the details of the analysis for the two-dimensional problem of a rectangular elastic plate under given edge loads. The method involves in the first place the determination of a harmonic function $\Phi(\theta)$ which reduces to specified values θ on the boundary. The boundary condition for the biharmonic stress function χ can then be expressed in the form $\chi=0$, $\partial\chi/\partial\nu=-\partial\Phi(\theta)/\partial\nu$. By transforming the region within the rectangle to the region within a unit circle $\rho=1$, such a harmonic function is determined as a Fourier series of cosines. The biharmonic function χ is then expressed as a series of separate biharmonic functions $\chi_n^{(n)}$ in the form $\chi=\sum_{n=1}^{\infty}A_n\chi_n^{(n)}$ which can also be determined as a Fourier series of cosines. By comparing the values of $-\partial\Phi(\theta)/\partial\nu$ and $\partial\chi/\partial\nu$ on the boundary the coefficients A_n may be found. Having determined the biharmonic function χ the stresses $\sigma_x=\partial^2\chi/\partial y^2$, $\sigma_y=\partial^2\chi/\partial x^2$, $\sigma_{xy}=-\partial^2\chi/\partial x\partial y$ can be evaluated, and the author has tabulated the required numerical values of the derivatives $\partial\chi_n^{(n)}/\partial\rho$ and $\partial^2\chi_n^{(n)}/\partial\rho^2$ as far as $n=6$, for points along the axis of y. These enable him to evaluate σ_x along the axis of y which he has graphed, and his results are in close agreement with those of other authors.

R. M. Morris (Cardiff).

Rothman, M. Isolated force problems in two-dimensional elasticity. II. Quart. J. Mech. Appl. Math. **3**, 469–480 (1950).

In an earlier paper [same vol., 279–296 (1950); these Rev. **12**, 371], the author considers plane strain and generalized plane stress, determining Stevenson's complex potential functions when there is concentrated loading by forces and couples. In the present paper this theory is applied to the circular disk with concentrated loads on its edges, and to a curvilinear polygonal disk loaded similarly. *G. E. Hay.*

Adem, José. An elementary solution of a problem of anisotropic elasticity. Bol. Soc. Mat. Mexicana **6**, 27–31 (1949).

The author treats the problem of a semiinfinite anisotropic plate occupying the region $x\geqq 0$, whose axes of symmetry coincide with the Cartesian axes x, y, z, and which is subjected to a normal concentrated force at the origin. The solution is obtained by elementary methods and leads to the conclusion that $\sigma_\theta=\tau_{r\theta}=0$, as in the classical case of isotropy.

A. W. Saénz (Washington, D. C.).

Uflyand, Ya. S. On the solution of the problem of bending of rectangular and sectorial plates for certain boundary conditions. Doklady Akad. Nauk SSSR (N.S.) **74**, 437–439 (1950). (Russian)

The solution of the problem is written in the form $u(x, y)=\sum_{n=0}^{\infty}u_n(y)\cos\alpha_n x$ and is shown to satisfy automatically simple support conditions at one edge of the plate, arbitrary conditions at the two edges adjacent to it, and one of the two fixed-edge conditions at the remaining edge. In order to satisfy the remaining (zero deflection) condition, a certain integral equation must be solved to determine the load required at the fixed edge. The expression for this load is then substituted into the plate equation together with the expression for the solution; $u_n(y)$ are next obtained by quadratures. *H. I. Ansoff* (Santa Monica, Calif.).

Polya, George, and Weinstein, Alexander. On the torsional rigidity of multiply connected cross-sections. Ann. of Math. (2) **52**, 154–163 (1950).

Consider a cylinder whose cross-section is multiply connected, i.e., is bounded by an outer curve and by several inner curves which include holes. The main result of the present paper is the following general theorem: Of all multiply connected cross-sections with a given area and with given joint area of the holes, the ring bounded by two concentric circles has the maximum torsional rigidity. The theorem generalizes a celebrated proposition due to Saint Venant, which corresponds to the extreme case when there are no holes. Saint Venant's statement has been proved only recently, by the method of symmetrization [Pólya and Szegö, Amer. J. Math. **67**, 1–32 (1945); Pólya, Quart. Appl. Math. **6**, 267–277 (1948); these Rev. **6**, 227; **10**, 206]. The proof of the main result is also based on the method of symmetrization, and it is preceded by a careful survey of the whole problem of torsion for multiply connected domains. In order to illustrate the theorem, the upper bound $(q^4-p^4)/2\pi$ is obtained for the torsional rigidity S of a frame whose inner boundary is a square of side p and whose outer boundary is a concentric square of side q. This upper bound for S is slightly better than the upper bound $\frac{1}{6}(q^4-p^4)$ (the polar moment of inertia of the frame) which was given by Diaz and Weinstein [Amer. J. Math. **70**, 107–116 (1948); these Rev. **9**, 480] who gave both upper and lower bounds for S.

J. B. Diaz (College Park, Md.).

Džanelidze, G. Yu. On the theory of thin or thin-walled bars. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 597–608 (1949). (Russian)

This paper presents a generalized theory of elastic deformation of thin-walled cylinders of uniform cross-section. The strain equations are taken in the form $e_x=e_y=\gamma_{xy}=0$,

$$\gamma_{xz}=(\partial\theta/\partial z)(\partial\phi/\partial x-y),\quad \gamma_{yz}=(\partial\theta/\partial z)(\partial\phi/\partial y+x),$$
$$e_z=e+\nu_1 y-\nu_2 x+\phi(x, y)(\partial^2\theta/\partial z^2),$$

where the terms not found in the classical treatment of the problem are ν_1 and ν_2 which are the curvatures of the axis of the shell and $\partial^2\theta/\partial z^2$, the derivative of the angle of twist. Energy expressions are then used to obtain the formulas for the applied moments and forces as well as for the natural boundary conditions. These formulas take account of the

interaction between torsion, bending, and extension which is neglected in the classical formulas. Equations of motion of a cylindrical shell in combined torsion and flexure are derived by use of Hamilton's principle. They differ from the corresponding classical equations by a term in the fourth derivative of the angle of twist. The results of the present theory are shown to be a generalization of Vlasov's work on thin-walled cylinders with open contours and Umanski's work on closed cylinders. *H. I. Ansoff.*

Klitchieff, J. M. Beams on elastic supports and on cross-girders. Aeronaut. Quart. 2, 157–166 (1950).

Continuing his interest in the use of trigonometric series for the solution of problems of built-up structures, the author treats the title problem by expanding all pertinent functions, such as beam deflections and bending-moment distributions, in terms of series of this type. This leads to various simplifications in the calculation of beams with simply supported ends and with discrete, elastic supports spaced along their length, as well as in the calculation of structures composed of groups of such elements. The elastic support parameters are arbitrary and the approach does not entail the use of concepts of beams on elastic foundations. When applied to the problem of beams supported by cross-girders, the work reduces to the solution of a system of linear algebraic equations of the same number as the number of cross-girders. *M. Goland* (Kansas City, Mo.).

Thomson, William T. The equivalent circuit for the transmission of plane elastic waves through a plate at oblique incidence. J. Appl. Phys. 21, 1215–1217 (1950).

L'auteur examine le problème de passage des ondes planes elastiques obliques à travers une lame plane se trouvant dans un milieu fluide de composition différente de chaque côté de lame. Pour les besoins de calcul la lame est remplacée par un circuit équivalent avec des impédances fonctions de l'angle d'incidence. L'auteur utilise les résultats obtenus par Reissner [Helvetica Phys. Acta 11, 140–155 (1938)] et par Smyth and Lindsay [J. Acoust. Soc. Amer. 16, 20–25 (1944)] dans des cas plus simples. En particulier, il emploie deux expressions auxiliaires M, N analogues à celles introduites par Reissner, ce qui lui permet de calculer les coefficients de transmissions et de réflexion de la lame. Une valeur de l'angle d'incidence est critique pour M et N et correspond à l'angle critique de l'onde de dilatation. La comparaison des courbes de transmission calculée et observée montre une concordance qualitative suffisante. *V. A. Kostitzin* (Paris).

Volterra, Enrico. Vibrations of elastic systems having hereditary characteristics. J. Appl. Mech. 17, 363–371 (1950).

Internal dissipation of energy in materials can be represented by viscous damping (stress-strain relation $\sigma=E\epsilon+A\dot{\epsilon}$) or by hereditary effects $(\sigma=f(\epsilon)+\int_0^t\varphi(t-\tau)(d\epsilon(\tau)/d\tau)d\tau)$. Energy dissipation in a single degree of freedom vibrating system having these characteristics is considered. The hereditary theory is shown to include the other if φ is chosen appropriately. Experimental dynamic stress-strain curves for rubberlike materials are reproduced for simple choices of φ. The assumption of special forms for φ permits its determination from the forced oscillation of a single degree of freedom system. Transverse free and forced oscillation of a simply supported beam are considered for material having a φ of exponential form. *E. H. Lee.*

Kovelites, James S. Free longitudinal vibration of a prolate ellipsoid, clamped centrally. Quart. Appl. Math. 9, 105–108 (1951).

The author extends the well-known solution for the free longitudinal vibrations of constant cross-section bars to the problem given by the title. The new equation of motion is then

$$\frac{\partial^2\xi}{\partial x^2}-\frac{2x}{a^2-x^2}\frac{\partial\xi}{\partial x}=\frac{\rho}{k}\frac{\partial^2\xi}{\partial t^2},$$

where ξ is the longitudinal displacement, ρ the volume density, k the modulus of elasticity, and $2a$ the length of the bar. After separating the variables by writing $\xi=X(x)T(t)$, the equation in X is solved by means of a power series. Expressions are given for the first three resonant frequencies. *H. D. Conway* (Ithaca, N. Y.).

Klotter, K. Die Biegeschwingungen eines Stabes unter pulsierender Achsialkraft bei beliebigen Randbedingungen. Ing.-Arch. 18, 363–369 (1950).

The author considers the stability of a beam acted upon by a periodic axial force $P_0+S\cos\Omega t$, where P_0, S, and Ω are constants, and t is the time. The case when the beam is simply supported at both ends has been treated by Lubkin and Stoker [Quart. Appl. Math. 1, 215–236 (1943); these Rev. 5, 83]. This case leads to Mathieu's equation, and the stability of the system is deduced from known properties of Mathieu functions. In the present paper a variety of end conditions are considered, including built-in conditions. The transverse deflection $y(x,t)$ is determined as an infinite sum of products of Mathieu functions of t and trigonometric and hyperbolic functions of x, with a quadruply infinite set of constant coefficients to be determined so that four functions of t vanish identically. The actual determination of these constants is not carried out, but an approximate determination of them is indicated. Known properties of Mathieu functions again permit a determination of the stability of the system. *G. E. Hay* (Ann Arbor, Mich.).

Lurie, Harold. A note on the buckling of struts. J. Roy. Aeronaut. Soc. 55, 181–184 (1951).

The author points out that if the equation of a buckled strut is written as a fourth order equation in the deflection rather than a second order equation in the bending moment, four end conditions can be satisfied instead of two in the latter case. For the column with built-in ends, it is shown that the first mode of buckling is the same for both equations, but the second critical load of the fourth order equation is lower than the corresponding load found from the second order equation. *G. H. Handelman.*

***Federhofer, Karl. Dynamik des Bogenträgers und Kreisringes.** Springer-Verlag, Vienna, 1950. xii+179 pp. $5.50.

This book is concerned with the natural frequencies and modes of vibration of curved girders and circular rings, with particular emphasis on members with thin walls and open sections. For particular cases the numerical results are demonstrated graphically and in tables, and where available corresponding experimental results are given as a check. When the exact analysis becomes too involved, approximate methods such as the Rayleigh-Ritz method are employed. Cross-sections of unsymmetrical form are considered, and also symmetrical sections with principal axes inclined to the plane of the curved center line, so that bending, twisting, and extensional vibrations may be coupled. Section dimen-

sions are considered small compared with the radius of curvature. The analysis is based on the general expressions for strain energy and kinetic energy including warping of the section due to torsion, the equations of motion being determined by variational methods. Warping of the section in its own plane is neglected, and the influence of shear distortion is considered only in special cases. Special cases treated in detail are simply symmetrical and doubly symmetrical cross-sections with a principal axis in the plane of curvature. Straight beams are considered, and also the influence of varying section, static pressure, a torus with internal pressure, and a beam with a parabolic center line.
E. H. Lee (Providence, R. I.).

Schindler, A. Essai de justification d'une détermination dynamique de fréquences et fonctions propres associées d'un avion et d'un calcul des masses généralisées correspondantes. Recherche Aéronautique **1950**, no. 15, 15–18 (1950).

This is an excerpt from another publication by the author [O.N.E.R.A. Publ. **43** (1950)]. When at different points of an airplane resonance curves are taken, the maxima occur, on account of the damping, at slightly different frequencies. The author tries to give an unambiguous definition of the natural frequencies and modes in this case and describes their experimental determination. The dynamic loads associated with the modes are calculated in a very abstract mathematical manner. *W. H. Muller* (Amsterdam).

Klouček, C. V. Structural analysis by distribution of deformation. Quart. Appl. Math. **9**, 77–88 (1951).

A discussion is given of the distribution-of-deformation method of structural analysis as well as of other similar methods for analyzing redundant structures. The convergence of the methods is considered, using as examples structures having two, three, and many joints. The order in which successive joints are given deformation increments is varied in the convergence analysis. The author points out that solutions for simultaneous loading, closed structures, joint displacement, etc. are given in earlier publications.
S. Levy (Washington, D. C.).

Allen, D. N. de G., and Sopwith, D. G. The stresses and strains in a partly plastic thick tube under internal pressure and end-load. Proc. Roy. Soc. London. Ser. A. **205**, 69–83 (1951).

A solution is given assuming the Tresca yield limit of constant maximum shear stress, and the deformation type stress-strain relation of proportionality between stress and total strain deviators. The development is in a convenient form for the analysis of problems with different end loads. Other solutions are reviewed, and detailed comparison is made with the solution based on an incremental type law [Hill, Lee, and Tupper, same Proc. Ser. A. **191**, 278–303 (1947); these Rev. **9**, 218]. For a closed end the solution is shown to apply to a high degree of approximation for the Mises flow limit, but not for an open end. No reference is made to the pertinent work of Hodge and White [J. Appl. Mech. **17**, 180–184 (1950); these Rev. **12**, 562].
E. H. Lee (Providence, R. I.).

Gubkin, S. I. Diagrams for regimes of mechanical states. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1950**, 1165–1182 (1950). (Russian)

The author's summary is as follows: "Two types of graphical representation of mechanical states are proposed which permit a classification of mechanical states, the mechanism of plastic deformation, and of the principal methods of metal shaping by pressure. A proposal is made to separate the mechanisms of plastic deformation into two classes: mechanisms with diffusion; and mechanisms free of diffusion. Certain properties are pointed out for each class."
H. I. Ansoff (Santa Monica, Calif.).

Hudson, G. E. A theory of the dynamic plastic deformation of a thin diaphragm. J. Appl. Phys. **22**, 1–11 (1951).

The author considers the problem of determining the plastic deformation of a thin metal diaphragm. The problem of solving the exact nonlinear plastic differential equations for the problem appears to be very difficult. Hence, the author develops a simplified theory. This theory is based upon the fact that due to the stresses acting on the diaphragm, a bending wave moves radially along the material and tilts it into a series of truncated conical rings. By use of the geometry of the bending wave and the equations of motion in terms of the stresses, the author obtains six relations for six independent functions of the theory. It is shown that there is no stress discontinuity at the wave front. Next, the author considers the thinning of the diaphragm. By use of appropriate plastic stress-strain laws, the author is enabled to check some of the formulas of this section derived by physical reasoning. Finally, the initial conditions for the various variables are discussed. Two methods of solution are considered: (1) an approximation scheme; (2) exact solutions for the case of no work hardening. The approximation method indicates the correct order of magnitude for the swing time. The more exact solutions give better approximations to the distance traveled by the deformation wave and the thinness of the diaphragm. *N. Coburn.*

Yagn, Yu. I., and Tarasenko, E. N. An applied theory of plastic deformation of beams. Doklady Akad. Nauk SSSR. (N.S.) **73**, 471–474 (1950). (Russian)

The formulas for the strain components obtained by the semiinverse method for elastic deformation are assumed to hold in the plastic range. The cross-section of the beam is assumed to be entirely in the plastic state, and the stress-strain relations are of the deformation type. Solutions are given for a symmetric beam for two simple cases of loading.
H. I. Ansoff (Santa Monica, Calif.).

Seide, Paul, and Stowell, Elbridge Z. Elastic and plastic buckling of simply supported solid-core sandwich plates in compression. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **967**, 10 pp. (1950).

Oldroyd, J. G. Rectilinear flow of non-Bingham plastic solids and non-Newtonian viscous liquids. II. Proc. Cambridge Philos. Soc. **47**, 410–418 (1951).

[For part I see the same Proc. **45**, 595–611 (1949); these Rev. **11**, 284.] The flows indicated in the title are caused by the given motion of the boundaries in the absence of a pressure gradient. The basic equations are found to be closely analogous to those governing steady two-dimensional irrotational subsonic flows of a gas with an arbitrary pressure-density relation. This analogy suggests the use of a Legendre transformation to linearize the problem. The flow caused by the axial motion of an infinite elliptic cylinder in an infinite plastic solid is treated as an illustration of the method. *W. Prager* (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

***Tudorovskiĭ, A. I. Teoriya optičeskih priborov. I. Obščaya čast. [Theory of Optical Instruments. I. General Part].** 2d ed. Izdat. Akad. Nauk SSSR., Moscow-Leningrad, 1948. 661 pp.

Fundamental laws of geometrical optics. Light energy; fundamental notions of photometry and colorimetry. The plane mirror and system of plane mirrors. Refraction through a plane and a system of planes. Resolution of light in refraction. Refraction through a spherical surface and a system of spherical surfaces; reflexion from spherical surfaces. Trigonometric computation of the passage of rays in a centered optical system. Theory of an ideal optical system. Limitation of beams in optical systems. Images given by real optical systems and their errors. Theory of aberrations of the third order. Chromatism of optical systems. Eiconals. Diffraction theory of images. *Table of Contents.*

Fok, V. A. Generalization of reflection formulas to the case of reflection of an arbitrary wave from a surface of arbitrary form. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 961–978 (1950). (Russian)

Sia g_{ik} ($i, k=1, 2$) la metrica della superficie riflettente, ω la fase dell'onda incidente nei vari punti di essa, s la distanza (lungo un raggio che abbia angolo di incidenza ϑ) della superficie riflettente da una data superficie d'onda riflessa, G_{ik} i coefficienti della seconda forma quadratica fondamentale della superficie riflettente. Si costruisca il tensore simmetrico $T_{ik}=g_{ik}-\omega_{,i}\omega_{,k}+s(\omega_{,ik}-G_{ik}\cos\vartheta)$. Se si chiama $D(s)$ il rapporto fra un elemento della superficie d'onda riflessa e il corrispondente elemento della superficie riflettente, risulta $D(s)\cos\vartheta=\det T_k^i$. Dalle leggi dell'ottica geometrica si ha poi che l'ampiezza sulla superficie riflettente e l'ampiezza sull'onda riflessa stanno fra loro nel rapporto $[D(s)/D(0)]^{\frac{1}{2}}$. La conoscenza di $D(s)$ e dei coefficienti di riflessione di Fresnel risolve dunque il problema.

G. Toraldo di Francia (Firenze).

Harris, Clyde W. Optical design by a matrix method. J. Opt. Soc. Amer. **40**, 819–822 (1950).

Si scelga un piano di riferimento, per esempio tangente al vertice di una superficie rifrangente del sistema, e sia **a** il vettore che determina in tale piano il punto d'intersezione di un raggio generico. Sia **s** un vettore parallelo al raggio e avente per modulo l'indice di rifrazione; sia **b** la proiezione di quest'ultimo vettore sul piano di riferimento. Si considerino i seguenti dieci vettori ($\mathbf{a}$, $\mathbf{b}$, $(\mathbf{a}^2)\mathbf{a}$, $(\mathbf{a}\cdot\mathbf{b})\mathbf{a}$, $(\mathbf{b}^2)\mathbf{a}$, $(\mathbf{a}^2)\mathbf{b}$, $(\mathbf{a}\cdot\mathbf{b})\mathbf{b}$, $(\mathbf{b}^2)\mathbf{b}$, $\delta\mathbf{a}$, $\delta\mathbf{b}$), dove gli ultimi due vettori rappresentano le variazioni dovute alle differenze cromatiche. L'autore istituisce una matrice quadrata del decimo ordine, che dà la trasformazione di questi vettori, quando il raggio passa da un piano di riferimento a un altro, e una matrice analoga, che dà la trasformazione corrispondente alla rifrazione attraverso a una superficie del sistema ottico. La trasformazione complessiva dal raggio incidente al raggio emergente dal sistema ottico risulta dal prodotto di tutte le matrici corrispondenti ai successivi intervalli e alle successive rifrazioni e dà contemporaneamente gli elementi parassiali, le aberrazioni del terzo ordine e le aberrazioni cromatiche.

G. Toraldo di Francia (Firenze).

Seman, O. I. The optical power of short electron lenses. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **20**, 1180–1193 (1950). (Russian)

La formula per il calcolo delle lenti elettroniche sottili

$$f^{-1}=\tfrac{1}{8}(\varphi_1)^{-\frac{1}{2}}\int_{z_0}^{z_1}(\varphi'^2/\varphi^{\frac{3}{2}})dz$$

viene dedotta ammettendo che nell'ambito della lente la distanza r del raggio dall'asse sia costante. In pratica questa formula può condurre ad errori anche del 100%. L'errore può essere notevolmente ridotto se si ammette invece che sia costante l'espressione $R=r\varphi^{\frac{1}{4}}$. Si può giungere così alla formula

$$f^{-1}=\tfrac{3}{16}(\varphi_a/\varphi_b)^{\frac{1}{4}}\int_{z_0}^{z_1}(\varphi'/\varphi)^2dz$$

dove gl'indici a e b si riferiscono agli spazi oggetto ed immagine rispettivamente. L'autore stima che con questa formula non si superi un errore dell'ordine del 10% anche per lenti spesse. *G. Toraldo di Francia* (Firenze).

Glaser, W. Richtungs-Doppelfokussierung von Elektronenbahnen in inhomogenen elektrisch-magnetischen Feldern. Österreich. Ing.-Arch. **4**, 354–362 (1950).

A thin electron beam in circular shape such as used in mass spectroscopy is considered. With fields symmetrical with respect to the axis and normal to the plane of the circular beam, the author examines electron paths close to the quasistatic circular electron path and establishes two orthogonally intersecting conical surfaces with apeces on the axis as the loci of the stable electron paths. In general, focusing of these two groups of electron paths will be at different angles leading to an astigmatic focus. For certain field conditions, the two angles become identical, double focus takes place with anastigmatic image formation. Several special cases are then treated as illustrations, such as the electrical cylinder condenser, the homogeneous magnetic field, the magnetic dipole field, and the combination of the last two. The utilization of these relations for mass spectroscopy is pointed out. *E. Weber.*

Glaser, Walter, und Bergmann, Otto. Über die Tragweite der Begriffe "Brennpunkte" und "Brennweite" in der Elektronenoptik und die starken Elektronenlinsen mit Newtonscher Abbildungsgleichung. Z. Angew. Math. Physik **1**, 363–379 (1950).

Affinché le definizioni matematiche di fuoco e di distanza focale abbiano utilità pratica l'autore ritiene necessario che il campo sia tale da ammettere traiettorie rettilinee parallele all'asse nello spazio oggetto o immagine rispettivamente. La condizione necessaria e sufficiente affinché ciò si verifichi, per esempio per lo spazio immagine, è che converga l'integrale $\int^\infty zQ(z)dz$ dove z è la coordinata assiale e $Q(z)$ dipende dal potenziale elettrico Φ e dall'iduzione magnetica B secondo la relazione $Q(z)=\frac{3}{16}(\Phi'/\Phi)^2+eB_z^2/8m\Phi$. L'analoga condizione per l'esistenza del piano focale, anziché del solo fuoco, è che converga l'integrale $\int^\infty z^2Q(z)dz$. Nel caso generale l'autore insiste sull'utilità della trasformazione parassiale osculatrice, cioè di quella trasformazione che effettivamente si realizza per un dato piano oggetto e per il suo piano immagine. *G. Toraldo di Francia.*

Rabin, B. M., and Straškevič, A. M. Trajectories of charged particles deflected slightly from their original direction by an electrostatic field. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **20**, 1232–1240 (1950). (Russian)

Calcolo di prima approssimazione delle traiettorie di particelle cariche in un campo elettromagnetico privo di speciali simmetrie, nel caso che le traiettorie stesse possano considerarsi quasi rettilinee. Valutazione dell'errore commesso nell'approssimazione. *G. Toraldo di Francia.*

Størmer, Carl. Résultats des calculs numériques des trajectoires des corpuscules électriques dans le champ d'un aimant élémentaire. VII. Trajectoires par l'origine faisceaux supplémentaires. Skr. Norske Vid. Akad. Oslo. I. **1949**, no. 2, 75 pp. (1949).

Part VI appeared in the same Skr. **1947**, no. 1; these Rev. **10**, 223.

Borel, Émile. Sur l'emploi des coordonnées de la droite pour l'étude des radiations. C. R. Acad. Sci. Paris **232**, 1329–1331 (1951).

The line coordinates are a, b, p, q, where $x=az+p$, $y=bz+q$; the corresponding infinitesimal element is

$$d\omega=da\,db\,dp\,dq\,(1+a^2+b^2)^{-2}.$$

The author shows how these formulae may be used to find, in a very simple manner, such quantities as (1) the quantity of energy passing a plane area in unit time, (2) the volume-density of energy inside a closed surface inside which a homogeneous isotropic radiation is taking place, and (3) the probable value of a line-function $\varphi(a, b, p, q)$ defined over a system of lines. *F. V. Atkinson* (Ibadan).

Epheser, H., und Schlomka, T. Flächengrössen und elektrodynamische Grenzbedingungen bei bewegten Körpern. Ann. Physik (6) **8**, 211–220 (1950).

The authors establish in two ways boundary conditions similar to those of Schlomka [same Ann. (6) **5**, 190–196, (1949); these Rev. **11**, 761]. For the first method they develop the Lorentz transformation formulae for the surface-normal and the surface charge and current densities, these being apparently new. The second, and simpler, method uses tensor calculus in four dimensions. The work is needed for a forthcoming paper on the electrodynamics of moving bodies. *F. V. Atkinson* (Ibadan).

Saunders, W. K. Uniqueness of solution of the exterior problem of the electromagnetic field. University of California Department of Engineering, Antenna Laboratory, Issue No. **175**, ii+8 pp. (1950).

This paper is concerned with the uniqueness of time-periodic solutions of Maxwell's equations in the homogeneous space outside a finite closed surface F which contains all sources in its interior. Given the values of the tangential electric field on F, the author proves that there exists at most one solution that satisfies $\lim_{r\to\infty} r\{(\mathbf{r}/r)\times\mathbf{H}+(\epsilon/\mu)^{\frac{1}{2}}\mathbf{E}\}=0$. This condition is discussed in detail and compared with Sommerfeld's "Austrahlungsbedingung" for scalar wave functions. *C. J. Bouwkamp* (Eindhoven).

Roubine, Élie. Sur le calcul du champ créé par un circuit en hélice. C. R. Acad. Sci. Paris **232**, 221–222 (1951).

The author has found expressions for the electromagnetic field produced by a perfectly conducting infinitely thin helix at points along the axis of the helix. *R. Phillips.*

Grün, Otto. Berechnung des elektrischen Feldes bei einer gewissen Materialverteilung. Math. Nachr. **4**, 419–433 (1951).

The Cartesian three-dimensional space is divided into 5 subspaces by 4 planes

$$z=a_1,\ z=a_2,\ z=a_3,\ z=a_4 \quad (0<a_1<a_2<a_3<a_4),$$

each of them having different electromagnetic constants. In each of these 5 subspaces there exists a Hertzian dipole located on the z-axis. The frequencies of all 5 Hertzian dipoles are equal, but not necessarily the intensities and the axes of polarization. The vector potential of the field of radiation for this array is given in the form of an integral expression. The general results are applied to the special case of one single dipole on the z-axis in the subspace $a_2\leqq z\leqq a_3$ with the axis of polarization parallel to the z-axis. *F. Oberhettinger* (Pasadena, Calif.).

Lucke, Winston S. Electric dipoles in the presence of elliptic and circular cylinders. J. Appl. Phys. **22**, 14–19 (1951).

By means of the vector Green's function for the infinitesimal electric dipole [J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1945, p. 466] satisfying the wave equation with the three-dimensional Dirac delta function as inhomogeneous term, the field vectors can be expressed as volume and surface integrals. The Green's functions are in turn found in the circular and elliptical cylinder coordinates and given as integrals in the complex plane. For the electric field it is assumed that its Green's function vanish on the surface of the cylinder; for the magnetic field, that the normal derivative of its Green's function so vanish. To make the complex integrals manageable, only the far field is considered with the appropriate asymptotic expressions for the Bessel and elliptic cylinder wave functions. Final results with polar graphs are given for the two major orientations of the dipole, namely, parallel to the axis and in a plane perpendicular to the axis of the cylinder which is assumed to be infinitely long. *E. Weber* (Brooklyn, N. Y.).

Silver, Samuel, and Saunders, William K. The external field produced by a slot in an infinite circular cylinder. J. Appl. Phys. **21**, 153–158 (1950).

The problem of the determination of the external field produced by an arbitrarily shaped slot in the wall of a circular wave guide of infinite extent and conductivity of its wall is handled as follows: The distribution of the tangential components of the electric field in the slot are supposed to be prescribed functions (for this purpose reasonable assumptions depending on the configuration of the slot are made). By the aid of these prescribed functions the tangential components of the electric field over the surface of the cylindrical wave guide can be given in the form of a Fourier series. On the other hand the Fourier expansions (with unknown coefficients) of the tangential components of the external field can be expressed as a superposition of a basic set of cylindrical waves. The comparison of both expressions on the surface of the cylinder determines the hitherto unknown Fourier coefficients of the external field and the problem is thus solved. Expressions for the far zone pattern and two examples (a) the slot runs completely around the cylinder with a constant (small) height, (b) the narrow rectangular slot parallel to the cylinder axis, are treated. *F. Oberhettinger* (Pasadena, Calif.).

Silver, Samuel, and Saunders, William K. The radiation from a transverse rectangular slot in a circular cylinder. J. Appl. Phys. **21**, 745–749 (1950).

The general results of an earlier paper [see the preceding review] are applied to the special case of a transverse rectangular slot, i.e., a slot bounded by the planes $z_1=-\frac{1}{2}l$, $z_2=\frac{1}{2}l$; $\Phi_1=-\alpha$, $\Phi_2=\alpha$, where the z-axis is identical with the axis of the cylinder. The results are restricted to the far zone field and radiation patterns have been computed. They show an excellent agreement with experimental data.

F. Oberhettinger (Pasadena, Calif.).

Kotani, Masao, and Takahashi, Hidetoshi. Theoretical determination of proper frequencies of the resonant circuit of the cavity magnetrons. J. Phys. Soc. Japan **4**, 65–72 (1949).

The author gives a mathematical analysis of two types of cavity magnetron, each consisting of a central cylindrical cavity A with 2ν resonant cylindrical cavities B equispaced around the circumference, and joined to A by narrow slots. In the first type all the B cavities are of the same size, in the second they are alternately large and small. The dimensions of the slots are assumed to be small compared with the operating wavelength, so that the problem of the strong local field due to charges of opposite sign which appear periodically on both sides of each slot may be solved by electrostatic methods. When the electric field is transverse and the magnetic field longitudinal, two sets of expressions for the field components in each B cavity and in A can be obtained, one set valid in the immediate neighborhood of the slots, and the other in the intermediate region between slots. These are matched at the boundary of A to give the characteristic equation for the resonant frequencies. A qualitative discussion of the roots of this equation shows that there is one natural wavelength which may be very long compared with the overall dimensions of the magnetron.

M. C. Gray (Murray Hill, N. J.).

Kotani, Masao, and Takahashi, Hidetoshi. Numerical tables of functions useful for the calculation of resonant frequencies of a cavity magnetron. J. Phys. Soc. Japan **4**, 73–77 (1949).

This is a companion paper to the one reviewed above. The functions involved in the characteristic equation include series whose terms are ratios of Bessel functions. Numerical values of the sums of these series are tabulated for each type of magnetron when the total number of B cavities lies between 4 and 12. *M. C. Gray* (Murray Hill, N. J.).

Kihara, Taro. Approximate methods regarding electromagnetic waves in hollow pipes and cavities. J. Phys. Soc. Japan **2**, 65–70 (1947).

The author suggests the use of a variational principle to determine the natural frequencies of electromagnetic cavity resonators. Thus let E be a vector subject to div $E=0$ inside the cavity, and $E\times n=0$ at the boundary. Then define the volume integrals $J=\int E^2dv$, $D=\int(\text{curl } E)^2dv$, taken throughout the volume of the cavity. If D is minimized subject to $J=1$ the vector E so determined can represent natural oscillations in the cavity. Alternatively, E is determined to make D/J a minimum, and the values of these minima are the eigenvalues of the oscillations. To apply the method to a given cavity the author chooses simple polynomial expressions for the components of E which satisfy the defining equations, and then the value of D/J gives a first approximation to the eigenvalues in terms of the parameters defining the cavity boundary. Formulas are given for ellipsoidal and elliptic cylinder cavities and for semicoaxial cavities. *M. C. Gray* (Murray Hill, N. J.).

Eckart, Gottfried. La propagation d'ondes électromagnétiques dans des couches de faible hétérogénéité. C. R. Acad. Sci. Paris **232**, 1294–1296 (1951).

Brazma, N. A. A new solution of the fundamental problem of the propagation of electromagnetic phenomena in a bundle of wires. Doklady Akad. Nauk SSSR (N.S.) **76**, 41–44 (1951). (Russian)

In a previous paper [same Doklady (N.S.) **69**, 313–316 (1949); these Rev. **11**, 297] the author discussed the solution of the vector-matrix telegraphic equations

$$-\partial\mathbf{u}/\partial x=\mathbf{R}\mathbf{i}+\mathbf{L}\partial\mathbf{i}/\partial t,\quad -\partial\mathbf{i}/\partial x=\mathbf{G}\mathbf{u}+\mathbf{C}\partial\mathbf{u}/\partial t$$

with $\mathbf{u}(0,t)=\mathbf{u}_0=\text{const.}$, $\mathbf{u}(l,t)=0$, $\mathbf{u}(x,0)=0$, $\mathbf{u}_t'(x,0)=0$, where $0\leqq x\leqq l$, $0\leqq t<\infty$. He has now succeeded in finding a more explicit and more general form for the solution; the initial condition $\mathbf{u}_t'(x,0)=0$ is here replaced by the "more natural" condition $\mathbf{i}(x,0)=0$. *F. V. Atkinson.*

King, Ronold. Theory of collinear antennas. J. Appl. Phys. **21**, 1232–1251 (1950).

The author discusses the radiation properties of a three-element collinear array of identical linear antennas. The two fundamental problems analyzed in detail are the symmetric and antisymmetric cases; in the former it is assumed that each antenna is center driven by a discontinuity in the scalar potential such that the currents at the center of each element are equal; in the latter the currents at the centers of the outer elements are equal, but opposite in sign to that at the center of the middle element. These two solutions may be combined to solve other problems, such as a center-driven middle antenna with two parasites, or with each outer antenna center-loaded with a lumped impedance. The analysis follows the usual lines of the Hallén integral equation method, using a King-Middleton type of impedance parameter [Quart. Appl. Math. **3**, 302–335 (1946); **4**, 199–200 (1946); these Rev. **7**, 401, 535]. With certain simplifying assumptions, a single integral equation of the Hallén type is obtained for the current in the middle antenna, with a modified kernel to account for the coupling with the outer elements. The formulas are too complicated to summarize, but detailed calculations for antennas of total length $\frac{1}{2}\lambda$ are outlined, and curves are drawn of the input and transfer impedances and of the current distribution. A practical application to a center-driven three-element array with phase reversing stubs is also discussed. *M. C. Gray.*

Muhina, G. V. On the screening effect of a conducting layer distributed over the contact region of two media. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 302–316 (1950). (Russian)

Replacing the upper horizontal layer of finite thickness h and conductivity k by an ideal plane of conductivity hk it is possible to simplify considerably the computation of the potential due to this layer, the screening effect of which must be eliminated since the purpose of geophysical surveying with the aid of electric methods is the study of tectonic structures beneath the upper alluvial layer. In this paper the simplified problem is solved, the main result being the delimitation of the circular region around the source of current outside of which the error caused by the simplification is negligible. *E. Kogbetliantz* (New York, N. Y.).

Quantum Mechanics

Eden, R. J. The quantum mechanics of non-holonomic systems. Proc. Roy. Soc. London. Ser. A. **205**, 583–595 (1951).

The quantum theory corresponding to the classical theory of a preceding paper [same vol., 564–583 (1951); these Rev. **12**, 645] is developed. Corresponding to the nonintegrable principal function is a Schrödinger function with a nonintegrable phase, which, however, leads to no ambiguity in the interpretation. In the Heisenberg picture it is necessary to introduce a quasiprojection operator Q_M which depends on the history of the state upon which it operates. After an infinitesimal time a state vector developing under the Hamiltonian will violate the constraints if it satisfied them initially. At each instant Q_M then projects it back into the subspace M defined by all vectors which do satisfy the nonholonomic constraints. *H. C. Corben* (Pittsburgh, Pa.).

Biedenharn, L. A note on time reversal and the Dirac equation. Physical Rev. (2) **82**, 100 (1951).

Faure, Robert. Intégrale première du premier ordre en théorie de Dirac. Nécessité de l'opérateur $\partial/\partial t$ dans les intégrales premières dépendant du temps. Forme de l'opérateur intégrale première. C. R. Acad. Sci. Paris **232**, 1469–1471 (1951).

Rideau, Guy. Sur la conservation de l'énergie en mécanique quantique. C. R. Acad. Sci. Paris **232**, 1409–1411 (1951).

Rideau, Guy. Méthode pour l'étude des perturbations à durée limitée. C. R. Acad. Sci. Paris **232**, 1338–1340 (1951).

Allard, Georges. Un nouveau type de perturbation. J. Phys. Radium (8) **11**, 646–652 (1950).

The author proposes to add to Dirac's Hamiltonian for an electron a term which is linear in the pseudovector $P^i = \epsilon^{ijkl}\pi_j\pi_k\pi_l$, where $\pi_i = p_i + (e/c)A_i$. For an electron in a pure Coulomb field, $P^0 = 0$, $\mathbf{P} = -(Ze^2mh/icr^3)\mathbf{r}\times\mathbf{v}$ and gives rise to a change in the magnetic moment of the electron. The author studies the case of hydrogen at length and finds a change in the magnetic moment of the order of $0.25\mu_0$, whereas Schwinger's result of $0.00116\mu_0$ is in good agreement with experiment [L. Rosenfeld, Nuclear Forces, Interscience, New York, 1948, vol. 2, p. 438]. *A. J. Coleman* (Toronto, Ont.).

Trkal, V. The general Lorentz transformation of the Dirac wave function. Rozpravy II. Třídy České Akad. **59**, no. 32, 18 pp. (1949). (Czech)

Let $\psi' = \Lambda\psi$ be the transformation of the Dirac wave function ψ under the general Lorentz transformation $x_i' = a_{ik}x_k$. The author's attempt is to find Λ as a function of the coefficients a_{ik} and of Dirac's matrices γ^k. As is well known, Λ has to fulfil the relations (I) $\Lambda\gamma^1\gamma^2\gamma^3\gamma^4 = \pm\gamma^1\gamma^2\gamma^3\gamma^4\Lambda$ and (II) $a_{ik}\gamma^k = \Lambda^{-1}\gamma^i\Lambda$. In order to avoid adopting a special choice of the four matrices γ^k, the author considers them as a base of a group of hypercomplex numbers. Considered as a hypercomplex number, Λ can be written, in the case of positive sign in (I), as a linear combination of 8 elements of this group, which form a subgroup of the original group, and as a linear combination of the remaining 8 elements in the case of negative sign. Representing these 8 elements of the group by means of quaternions, the author is able, with use of the relation (II), to determine in general form all coefficients in the above mentioned linear combinations. The explicit calculation is carried out in the case of positive sign only. It turns out there exists an infinite number of mutually equivalent solutions, some of which are remarkable because of their relative simplicity. *M. Brdička* (Prague).

Lehmann, H. Zur Regularisierung der klassischen Elektrodynamik. Ann. Physik (6) **8**, 109–123 (1950).

The work of other authors [Feynman, Physical Rev. (2) **74**, 939–946 (1948); McManus, Proc. Roy. Soc. London. Ser. A. **195**, 323–336 (1948); these Rev. **10**, 222, 664] on convergent modifications of classical electrodynamics is continued. Previous results are surveyed from the point of view of possible modifications of the Green's functions of the wave equation. It is shown that there exists a formulation which achieves finiteness at the cost of bringing in negative energy particles which, however, are not radiated. Unfortunately, the fields in this theory are no longer purely retarded. This brings in the possibility of peculiar motions for electrons which are not investigated in detail here. The quantum difficulties associated with these modifications are not discussed. *K. M. Case* (Ann Arbor, Mich.).

Leite Lopes, J. Covariant canonical formalism of Maxwell's electrodynamics. Anais Acad. Brasil. Ci. **22**, 349–369 (1950). (Portuguese)

The Hamiltonian form of Maxwell equations is reviewed and it is noted that manifest covariance is lost in passing from the Lagrangian to Hamiltonian form. To remedy this a covariant Hamiltonian formulation is set up similar to those given by Kroll [Physical Rev. (2) **75**, 1321 (1949)] and Matthews [ibid., 1270 (1949)]. Covariant classical analogs of the Heisenberg and Schrödinger representations are achieved by means of covariant Poisson bracket relations and a covariant Hamilton-Jacobi equation. It is shown how classical point charges can be included in the theory. Lastly the modifications occasioned by Wentzel's λ-limiting process and McManus' finite electron are indicated. *K. M. Case* (Ann Arbor, Mich.).

Matthews, P. T. Renormalization of neutral mesons in three-field problems. Physical Rev. (2) **81**, 936–939 (1951).

The divergence difficulties arising when neutral spin zero mesons are added to a mixture consisting of nucleons, photons, and charged spin-zero mesons are considered. It is found that with scalar or pseudoscalar coupling a finite number of contact interactions can be added which cancel all divergences. These contact interactions can then be interpreted as renormalizations. With pseudovector coupling no such renormalization program is possible since the number of primitive divergences is unlimited. The possibility of using regulators when the renormalization program fails is considered. It is shown that essential ambiguities result. Hence it is concluded that regulators may only be used for those theories where renormalization is possible. *K. M. Case* (Ann Arbor, Mich.).

Dempster, J. R. H. Note on the relation between Feynman's formulation of scattering problems and the Born approximation. Canadian J. Physics **29**, 66–71 (1951).

The Born approximation treatment of scattering problems is compared with the space-time formulation recently given by Feynman [Physical Rev. (2) **76**, 749–759 (1949)]. It is

shown that the Born approximation Green's function satisfies the same differential equation as does Feynman's field free kernel when integrated over the time coordinate. By explicit computation it is shown that the successive terms in the Born approximation are identical with those of Feynman. From this it is concluded that the regions of validity are identical. *K. M. Case.*

Gupta, Suraj N. The S-matrix and radiation damping. Proc. Cambridge Philos. Soc. **47**, 454–456 (1951).

An alternative form of expansion of the S-matrix is given which is unitary to any order of approximation. The nth term, instead of containing the contributions of all nth order transitions, contains such of these as are not made up of two or more real transitions. *H. C. Corben.*

Blohincev, D. I. Elementary particles and fields. Uspehi Fiz. Nauk **42**, 76–92 (1950). (Russian)

A general expository article on the background of current theories of the relation of particle and field aspects of atomic physics. It is in three sections entitled: (1) What does quantum mechanics say about the nature of particles? (2) Particles arising from quantization of an harmonic oscillator field. (3) Particles and the principle of spectral decomposition. *A. J. Coleman* (Toronto, Ont.).

Slansky, Serge. Champ soustractif et énergie propre de l'électron. C. R. Acad. Sci. Paris **232**, 1191–1193 (1951).

Vigier, Jean-Pierre. Introduction géométrique des particules élémentaires en théorie unitaire affine. C. R. Acad. Sci. Paris **232**, 1187–1189 (1951).

Rumer, J. B. Physikalischer Inhalt der 5-Optik. Sowjetwissenschaft. Naturwiss. Abt. **1950**, no. 2, 96–103 (1950).

Translation of Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 199–205 (1950); these Rev. **12**, 226.

Thermodynamics, Statistical Mechanics

Longuet-Higgins, H. C. The statistical thermodynamics of multicomponent systems. Proc. Roy. Soc. London. Ser. A. **205**, 247–269 (1951).

A conformal solution is defined by the following conditions: (1) The mutual potential energy of a molecule of species r and another one of species s at a distance R is given by the expression $U(R)=f_{rs}u(g_{rs}R)$, where the f_{rs} and g_{rs} are constants characteristic of the pair r, s. (2) The f_{rs} and g_{rs} are close to unity. (3) $g_{rs}=\frac{1}{2}(g_{rr}+g_{ss})$. From these assumptions it is possible to calculate rigorously the thermodynamic properties of a conformal solution. The predicted relations between free energy, entropy, heat, and volume of mixing agree well with available data on nonpolar solutions. The theory makes no appeal to a lattice model, such as Fowler and Guggenheim's model, and can therefore be applied both to liquids and imperfect gases and to two-phase two-component systems near the critical point. *F. W. London* (Durham, N. C.).

Katsura, Shigetoshi, and Fujita, Hisaaki. Some remarks on the condensation phenomena. Progress Theoret. Physics **5**, 997–1009 (1950).

The system consisting of N molecules in a volume V is divided in m cells of equal size, each of volume V/m. The number of those cells which contain i molecules is denoted by m_i. Hence $\sum m_i=m$ and $\sum im_i=N$. If m_i, considered as a function of i, has two sharp maxima, the coexistence of two phases may be assumed. The authors, neglecting the interaction energy between two particles which belong to different cells, replace the integration of the partition function in $3N$-dimensional space by the summation, with regard to the sets of m_i, of the products of the integrals over the $3i$-dimensional cells. Replacing this sum by its largest term means picking up the most probable microcanonical ensemble. It is shown that the condition $dm_i/di=0$ has only one root for temperatures above a certain critical temperature T_c and three roots for temperatures lower than T_c. In the latter case m_i has two maxima and one minimum. *F. W. London* (Durham, N. C.).

Fraser, A. R. The condensation of a perfect Bose-Einstein gas. I, II. Philos. Mag. (7) **42**, 156–164, 165–175 (1951).

The present paper gives a rigorous derivation of the condensation of a perfect Bose-Einstein gas and confirms earlier results obtained by nonrigorous elementary methods by the reviewer [Physical Rev. (2) **54**, 947–954 (1938)] and dispels doubts raised as to the validity of these results by G. Schubert [Z. Naturforschung **1**, 113–120 (1946); **2a**, 250–251 (1947); these Rev. **8**, 556; **10**, 92], and others. Formulae are given for the mean occupation numbers of the energy levels and for their mean fluctuations. *F. W. London.*

Klein, M. J., and Tisza, L. Theory of critical fluctuations. Physical Rev. (2) **76**, 1861–1868 (1949).

The authors consider a macroscopic system as subdivided into cells of identical size and shape arranged in a regular spatial array. Instead of representing one of the cells by a canonical ensemble, treating the rest as a reservoir, they treat all cells on equal footing. This method is found appropriate to deal with the fluctuations near the critical point for which the standard theory yields infinite results. The method is general enough to include λ-points in solids. The macroscopic system is invariant under the group of translations which transplace one cell into another. The thermodynamic parameters are invariants of this group. *F. W. London* (Durham, N. C.).

Tisza, L. Theory of superconductivity. Physical Rev. (2) **80**, 717–726 (1950).

Localized atomic wave functions are used to construct many-electron wave functions obeying the exclusion principle and corresponding to definite electronic localization (ϕ-functions). Crystal translations will transform these into a set of ω equivalent ϕ-functions. By linear superposition of ϕ-functions, zero-order wave functions of the correct symmetry (ψ-functions) can be constructed. The model has superconducting properties if its lowest state has a ψ-function obtained through the superposition of at least three ϕ-functions of a small ω and if this state is depressed below the continuum of high ω-states. The present method is not adequate to prove that the conditions stated are actually satisfied in superconductors. *F. W. London.*

Luttinger, J. M. A note on Tisza's theory of superconductivity. Physical Rev. (2) **80**, 727–729 (1950).

Tisza [see the preceding review] has only shown that his model leads to the diamagnetic equations in superconductors. In the present paper the equations which connect the electric field with the current in superconductors are derived from the same assumptions. *F. W. London.*

Schachenmeier, R. Zur Quantentheorie der Supraleitung. Z. Physik 129, 1–26 (1951).

Heywang, Walter. Reflexionseffekte bei der nichtlinearen Theorie der Supraleitung. Ann. Physik (6) 8, 187–200 (1950).

Davydov, A. S. The theory of dispersion of molecular crystals in the infrared region. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 19, 930–936 (1949). (Russian)

The problem of dispersion of crystals is studied by using a one-dimensional model. Intramolecular vibrations and lattice vibrations are considered. The interaction between these two kinds of vibration leads to the appearance of a secondary maximum in the absorption curve, which loses the symmetrical form found for gases. The approximations should be good for infrared rays and very low temperatures, but the results may also have some relevance for other regions of the spectrum and higher temperatures.

W. H. Furry (Cambridge, Mass.).

Davydov, A. S. The theory of dispersion of molecular crystals. II. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 20, 760–766 (1950). (Russian)

The model considered is a one-dimensional lattice in which molecules can execute rotational oscillations around an equilibrium orientation, as well as having internal vibrational degrees of freedom. Because of the interaction between the two kinds of motion, the absorption curves are not given correctly by superposition of separate curves corresponding to individual degrees of freedom, but have more complicated shapes. *W. H. Furry.*

Davydov, A. S. On the theory of dispersion of molecular crystals. Izvestiya Akad. Nauk SSSR. Ser. Fiz. 14, 502–507 (1950). (Russian)

A one-dimensional lattice is used as a model for studying the dispersion of crystals. The considerations given resemble those in previous papers by the author [see the two preceding reviews]. The molecules are supposed capable of internal vibrations and rotatory oscillations, besides the lattice vibrations. Various curves are given for index of refraction and extinction coefficient. *W. H. Furry.*

BIOGRAPHICAL NOTES

✱Mehanika v SSSR za tridcat' let 1917–1947. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 416 pp.

This volume surveys the progress in mechanics in the USSR during the years 1917–1947. Each essay is accompanied by a rather extensive bibliography. The contents are as follows. General outline of the development of mechanics in Russia and in the USSR (N. D. Moiseev). General mechanics: Analytic dynamics (V. V. Dobronravov); Stability of motion (G. N. Dubošin); Oscillations (N. N. Bogolyubov); Applied theory of the gyroscope (Ya. N. Roĭtenberg); Theory of mechanisms (I. I. Artobolevskiĭ); Mechanics of bodies of variable mass (A. A. Kosmodem'yanskiĭ). Elasticity and plasticity: Three-dimensional problems of the theory of elasticity (G. S. Šapiro); The fundamental plane and contact (mixed) problems of the static theory of elasticity (D. I. Šerman); Plates and shells (Yu. N. Robotnov); Plasticity (A. Yu. Išlinskiĭ); New problems of structural mechanics (M. M. Filonenko-Borodič); Elastic waves (V. G. Gogoladze). Hydro-aeromechanics: Waves (L. N. Sretenskiĭ); Boundary layer (L. G. Loĭcyanskiĭ); Gas dynamics (I. A. Kibel'); Turbulence (A. M. Obuhov); Wing and propeller of an airplane (V. V. Golubev); Unsteady motion (L. P. Smirnov); Gliding (L. I. Sedov); Hydraulics (I. A. Čarnyĭ); Hydrodynamic theory of filtration (I. A. Čarnyĭ).

Acta Physica Academiae Scientiarum Hungaricae

This journal replaces Hungarica Acta Physica. Vol. 1, no. 1, is dated 1951.

VI Zjazd Matematyków Polskich.

The proceedings of the 6th Congress of Polish Mathematicians, Warsaw, Sept. 20–23, 1948, has been issued as a supplement to vol. 22 of the Ann. Soc. Polon. Math. It is dated Cracow, 1950. Since most of the papers are either expository or short abstracts, they will not, in general, be reviewed separately.

Archiv für mathematische Logik und Grundlagenforschung.

Vol. 1, no. 1, is dated September, 1950. The journal is published by W. Kohlhammer Verlag, Stuttgart.

Časopis pro Pěstování Matematiky a Fysiky.

Vol. 74 (1949), parts 2–4 is devoted to the proceedings of the joint 3d Congress of Czechoslovakian Mathematicians and 7th Congress of Polish Mathematicians held at Prague, 28 August–4 September, 1949. Only some of the longer papers will be reviewed in these Rev., the others being generally abstracts. Papers written in Czech or Polish are accompanied by abstracts in English, French, or German.

Izvestiya Akademii Nauk SSSR. Seriya Geografičeskaya i Geofizičeskaya.

Beginning with no. 2, 1951, this journal has split into two journals: Seriya Geografičeskaya and Seriya Geofizičeskaya. The last issue of the combined series was vol. 15, no. 1; the new journals carry no volume number.

Koninklijke Nederlandse Akademie van Wetenschapen, Proceedings.

Starting with vol. 54, the Proceedings have split into three series: Series A, Mathematical Sciences; Series B, Physical Sciences; Series C, Biological and Medical Sciences.

Proceedings of the Benares Mathematical Society.

This journal has been discontinued with vol. 9, no. 2. It has been replaced by a new journal, Gaṇita, published by Bhârata Gaṇita Pariṣad, Lucknow, India. Vol. 1, no. 1 is dated June, 1950.

Zeitschrift für angewandte Mathematik and Mechanik.

Heft 8/9 of Band 30 contains short reports of the papers given at the meeting of the Gesellschaft für angewandte Mathematik und Mechanik held on 16–19 April 1950 in Darmstadt. These papers will not be reviewed separately in Mathematical Reviews.

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